AZIMUTH LINE OF POSITION

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1. Introduction. If X and S are points of the earth's surface, the *azimuth* at S of X is the angle which the great circle SX makes with the north direction of the meridian of S. If X is in known position, an observer who finds that the azimuth of X at his position is θ may conclude that he is somewhere on the *isoazimuth* (X, θ), *i.e.*, on the locus of earth points from which the azimuth of X is θ. It will be shown below how to use θ and the position of X to find on a large scale map (which need not include X) a straight line closely approximating the part of (X, θ) on the map. Such a line will be called an *azimuth line of position* because the observer's position must lie close to that line [10].

The large scale map used may be any projection on which terrestrial distances and angles can be laid off with reasonable accuracy. Hence the technique to be described may be used with Mercator networks, either printed ones or ones specially prepared as occasion arises; with Lambert maps such as those of the United States Civil Aeronautics Administration or the French War Office; with American Polyconics such as those of the United States Coast and Geodetic Survey showing the coastal waters of the United States, or those on which the United States Army Military Maps are based; or with British Ordnance Polyconics.

If the Greenwich Civil Time is known, the latitude and longitude of the earth point directly under a heavenly body, be it sun, moon or star, may be determined from the nautical almanac [1], so that point will serve as an X. If both the altitude above the horizon and the azimuth of the heavenly body at a point S were accurately known, the position of S could be found by solving a spherical triangle. (The solution is in general ambiguous but there will, except when the difference of longitude of X and S is nearly 90°, be no doubt as to which solution gives S.) In practice the measurement of altitude by means of a sextant can be made accurately and hence a reliable line of position, the Sumner line, can be found from this measurement [1]. The azimuth on the other hand, because of mechanical difficulties in sighting and because its correctness depends on knowing the direction of true north accurately, cannot be found reliably enough to make a solution of the astronomical triangle worth while. But crossing the azimuth line of position to be described here with the Sumner line will give a graphical solution of the astronomical triangle which separates the part of the result which is accurate (the Sumner line) from the part depending on less accurate measurement (the azimuth). Direct computation by spherical trigonometry gives no such indication of the direction in which error in the fix is to be expected.

In the daytime when only the sun is visible, the use of an azimuth line of position thus makes it possible to get on a large scale map a fix, approximate in the sense described, without waiting the two or three hours necessary to cross one Sumner line, carried forward by dead reckoning, with another. (See example

1, section 5.) In general the azimuth line makes with a Sumner line simultaneously obtained a large enough angle. (See end of section 3.)

If X is a radio transmitter in known position its azimuth at S may be measured and an azimuth line of position plotted. Until the accuracy of methods for obtaining a radio azimuth is improved it is hardly worth while to plot the azimuth line on a large scale map since the standard small scale map approximations are adequate. The method now used of approximating an isoazimuth by a straight line (rhumb line) on a small scale Mercator [2] is however very rough at large distances and so could not long survive improvement in the technique for determining the azimuth. The approximation of an izoazimuth by a straight line (approximate great circle) on a Lambert Conformal map at present in use [2] would be still less satisfactory at long distances.

Like most calculations involving a spherical surface the ones needed here to find an azimuth line of position are not trivial. The work below is, however, so arranged that tables depending on three variables similar to the "Tables of Computed Altitude and Azimuth" (United States Hydrographic Office Publication no. 214) could be prepared were it ever thought worth while. Use of such tables would make the computations trivial.

A reader interested only in the application of the results of this paper may now skip to sections 4 and 5.

2. Map projections on which isoazimuths have special character. The considerations of this paper are all based on a spherical earth. The formula [3]

$$\cot A \sin C = \cot a \sin b - \cos C \cos b \tag{2.1}$$

holds for any spherical triangle ABC where a is the side opposite angle A, *etc.* if ϕ is the angular distance of a point S from the north pole, and ϕ_x is the angular distance of a fixed point X from the north pole, and λ is the difference of longitude of S and X, (2.1) shows that S is on (X, θ) if and only if

$$\cot \theta \sin \lambda = \cot \phi_x \sin \phi - \cos \phi \cos \lambda, \tag{2.2}$$

so that (2.2) is the equation of (X, θ) in spherical coordinates. From (2.2) it readily follows that the isoazimuths form a three parameter family on the earth, that all of them except the equator pass through both the north and south poles, that (X, θ) makes at X the angle θ with the southerly direction of the meridian of X if the angle is measured towards S.

In 1833 J. J. Littrow discovered a map projection whose theory was later applied to the study of isoazimuths by Captain Weir and H. Maurer [4]. If (x, y) are rectangular cartesian coordinates, the equations of the map are:

$$x = \sin \lambda \csc \phi, \qquad y = \cos \lambda \cot \phi. \tag{2.3}$$

Substituting (2.3) in (2.2) yields an equation linear in x and y. Hence isoazimuths (X, θ) for all θ and for points X on the meridian $\lambda = 0$ are represented by straight lines, and every straight line of the map represents such an isoazimuth. The map is conformal except at the points $\lambda = \pm 90°$, $\phi = 90°$ and is a

two sheeted Riemann surface with algebraic singularities at these points. If the complex z-plane is the Littrow map and the w-plane is a polar stereographic map, then $z=w+1/w$. The Littrow map represents the meridians and parallels by confocal hyperbolas and ellipses, the meridian $\lambda=0$ being a straight line. The construction here made of the azimuth line of position depends on the theory of this map.

For the sake of completeness four other special maps will now be described though no use is made of them in the subsequent sections. Maurer [5] has constructed a conformal map in which isoazimuths of stations on $\lambda=0$ are represented by circles. If two points, X and Y, on the equator are given differing by less than 90° in longitude, Maurer [6] has also shown how to construct a map such that (X, θ) and (Y, θ) are straight lines for every θ. This map is conformal at X and at Y but not generally.

Dilloway [7] has described the construction of a map for any two stations X and Y such that on it (X, θ) and (Y, θ) are all represented by straight lines and that the map permits the angles θ to be laid off directly at X and Y. At and near the stations and the line joining them Dilloway's map will represent different points of the earth, sometimes near together, by the same map point. In other words the Dilloway map has a complicated singular region right at its center.

If (ρ, ψ) are polar coordinates in the plane, I have found that a map given by the equations

$$\rho = 2 \cot \phi, \qquad \psi = \lambda \tag{2.4}$$

carries the isoazimuths $(X, 0)$ and $(X, 90°)$ for all X into the straight lines of the map. The map image of $(X, 90°)$ is orthogonal to the map meridian of X. Moreover on this map (X, θ) is asymptotic as $\rho \to \infty$ to those lines through X which make the map angles $\pm\theta$ with the southerly direction of the map meridian of X. Since the map is not conformal, this does not imply that the map image of (X, θ) is tangent at X to its asymptotes. The map is two sheeted, one sheet for the northern and one for the southern hemisphere (one asymptote is in each hemisphere) and represents the whole equator by the point $\rho=0$.

3. Azimuth line of position. Let ϕ be the angular distance of a point A from the north pole;

ϕ_x be the angular distance of X from the north pole;

λ be the difference of longitude of the points A and X;

θ_c be the azimuth of X at A;

θ_0 be the azimuth of X at an arbitrary point S.

In the sequel A will be an "assumed position" for which ϕ and λ are therefore known and for which θ_c and the distance XA can therefore be found by spherical trigonometry; θ_0 will be the "observed" azimuth at the unknown position S.

Let M be a Littrow map constructed for the meridian of X. Then on M the curve (X, θ_c) and (X, θ_0) are represented by straight lines q and t making the angle $\theta_0-\theta_c$ at the map image X' of X, q going through the map image A'

of A, and t through the image S' of S (figure 1). Let i be the inclination of the map meridian of A, *i.e.*, the angle that meridian makes at A with the meridian of X. Then $\tan i = (dx/dy)$ for constant λ and so using (2.3)

$$\tan i = \tan \lambda \cos \phi. \tag{3.1}$$

Hence the angle α that q makes with the meridian of A is given by

$$\alpha = \theta_c + i. \tag{3.2}$$

Let D' be the foot of the perpendicular from A' to the map meridian of X. Then on map M, using (2.3), $X'A' = A'D' \csc \theta_c = (\sin \lambda \csc \phi) \csc \theta_c$. But if P is

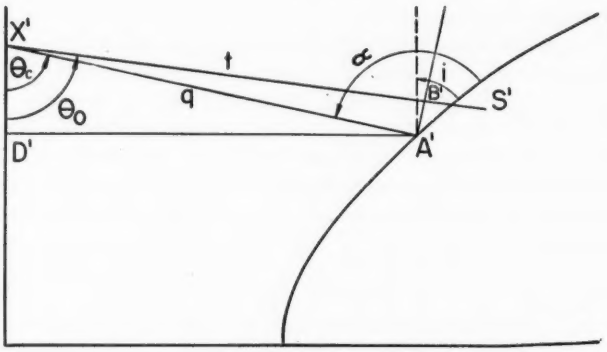

FIG. 1

the north pole and d is the great circle distance XA in angular units, the law of sines for the spherical triangle XPA gives $\sin \phi_x / \sin \theta_c = \sin d / \sin \lambda$ so

$$X'A' = \sin d \csc \phi \csc \phi_x. \tag{3.3}$$

On map M let the perpendicular at A' to q meet t in the point B'. Then $A'B' = X'A' \tan (\theta_0 - \theta_c)$ or, using (3.3)

$$A'B' = \sin d \csc \phi \csc \phi_x \tan (\theta_0 - \theta_c). \tag{3.4}$$

To approximate the true length r on the earth of the curve u corresponding to $A'B'$ it is necessary to divide (3.4) by the scale of the map M at A'. If ds and dS are arc length on map M and on the earth, then (2.3) yields

$$dS = \frac{10{,}800 \sin \phi}{\pi \sqrt{\cot^2 \phi + \cos^2 \lambda}} ds \text{ nautical miles.} \tag{3.5}$$

But if

$$\tan \psi = \cos \lambda \tan \phi, \quad \text{then} \quad \sqrt{\cot^2 \phi + \cos^2 \lambda} = \cot \phi \sec \psi. \tag{3.6}$$

So (3.4) and (3.5) yield

$$(3.7) \qquad r = \left[\frac{10{,}800}{\pi} \sin d \tan \phi \csc \phi_x \cos \psi\right] \tan (\theta_0 - \theta_c).$$

Notice that i, θ_c and the square bracket of (3.7) depend only on the positions of X and A and so could be tabulated in terms of λ, lat. A and lat. X in tables similar to H.O. 214.

Let K be a map on which in a small region terrestrial angles by direct measurement and (scaled) distances may be adequately approximated, and let A from now on be the map K image of A. Let n be a straight line on K making the angle α (towards X) with the map meridian at A. Then n is, by what has gone before, the tangent at A to the image on map K of (X, θ_c). Let p be a straight line on K perpendicular at A to n. For short distances p closely approximates the curve u of the earth. Let B be a point on p at (scaled) distance r from A. If $\theta_0 > \theta_c$ put B on the northerly side of A; if $\theta_0 < \theta_c$ on the southerly. (If p runs east-west put B east of A if and only if S is east of X as shown by θ_0.) Then B closely approximates the image on map K of B'. Let m be a line on B making the angle $\theta_0 - \theta_c$ with n in such a way that m and n, if produced, would meet on the X side of A. Then m closely approximates a tangent to (X, θ_0) and so is the required azimuth line of position.

In order that the line AB be short and the approximation good, it is necessary that A be not too far from S, so A should be near the dead reckoning position of S (see the examples, section 5). Since $\theta_0 - \theta_c$ is then small, the angle between an azimuth line m and a simultaneous Sumner line is about $90° - i$. Formula (3.1) shows under what conditions this is so small as to magnify the error unduly (see section 1).

4. Summary of directions for drawing azimuth line of position. Let K be a map on which terrestrial angles and distances may be laid off. Let L_x be the latitude and λ_x the longitude of a point X from which light or radio signals are received. As described in section 1, X may be either a terrestrial point under a bright heavenly object or a radio transmitter.

i. Determine the bearing θ_0 at your position of X, measuring θ_0 toward X from the north direction of your meridian.

ii. Choose near your dead reckoning position an assumed position A in lat. L_a, long. λ_a. Let λ be the difference of longitude of X and A.

iii. Let θ_c be the azimuth of X at A, again measured towards X, and let d be the great circle distance XA. These two quantities can be determined by solving the spherical triangle XAP where P is the north pole [8].

iv. On map K lay off a line n through A making the angle α (toward X) with the map meridian of A where

$$(4.1) \qquad \alpha = \theta_c + i$$

$$(4.2) \qquad \tan i = \pm \tan \lambda \sin L_a \begin{cases} + \text{ sign if } L_a \text{ is } N, \\ - \text{ sign if } L_a \text{ is } S. \end{cases}$$

Draw a line p through A perpendicular to n. Let ψ and r be such that:

$$\tan \psi = \cos \lambda \cot L_a \tag{4.3}$$

$$r = \frac{10{,}800}{\pi} \sec L_x \cot L_a \cos \psi \sin d \tan (\theta_0 - \theta_c) \text{ nautical miles.} \tag{4.4}$$

Locate a point B by laying off on p a length r from A in the northerly direction if $\theta_0 > \theta_c$, southerly if $\theta_0 < \theta_c$. If p runs east-west, put B east of A if and only if you are east of X. The direction of X will be shown by θ_0.

v. Through B draw a line m making the acute angle $\theta_0 - \theta_c$ with n in such a way that m produced and n produced would meet on the side of p at which X lies. Then m is your azimuth line of position.

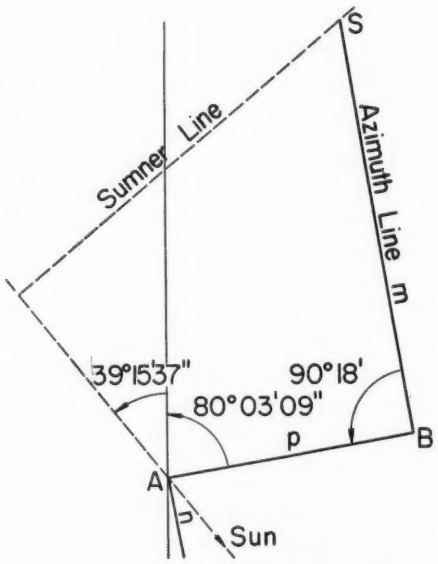

FIG. 2 (Scale: 1 inch to 14″ of Longitude)

5. Examples. The first example illustrates the obtaining of an approximate fix as the intersection of a Sumner line and an azimuth line of position based on simultaneous observations of the altitude and azimuth of the sun's center. In the second example a fix is obtained by crossing azimuth lines of position on two radio transmitters. All standard spherical trigonometry computations, including those for the Sumner line in example 1, are here omitted, only the results being given. The first column of the tabular computation refers to example 1, the second two columns refer to example 2, transmitters X and Y

respectively. The quantities to be plotted on map K are underlined. In figures 2 and 3 maps K are large scale Mercator networks, the longitude scales being as shown on each figure, the latitude scales then being obtained from the table of "Meridional Parts or Increased Latitudes" which is table 3 of the Bowditch "American Practical Navigator" (H.O. no. 9). Distances were plotted in the standard way by using the latitude scale.

The "observed" data were computed by spherical trigonometry from the true positions of the observers which are: in example 1, 42°28′N, 76°31′W; in example 2, 36°N, 60°W; and so, especially in example 2, they are much more accurately known than they would be in practice.

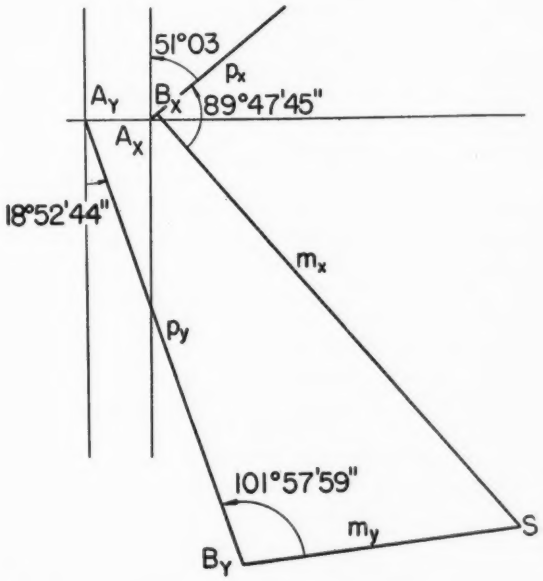

FIG. 3 (Scale: 1 inch to 1° of Longitude)

Example 1. On January 20, 1939 at Greenwich Civil Time $14^h38^m00^s$ an observer near Ithaca, N. Y. (42°26′27″N, 76°29′00″W), finds that the altitude of the sun's center is 17°15′40″ and the azimuth of the sun's center from his position is 140°44′55″ E of N. From these data determine the observer's position.

Example 2. A ship in dead reckoning position 38°12′N, 62°35′W finds that radio transmitter X on Mt. Desert Rock (43°58′08″N, 68°07′44″W) bears 35°27′46″W of N and that transmitter Y on Cape Hatteras (35°15′17″N, 75°31′16″W) bears 88°49′23″ W of N. From these data determine the ship's position.

In the first example one assumed position is used, in the second, two, A_x for transmitter X and A_y for transmitter Y. The coordinates of the assumed positions appear in the tabular computations.

		Example 1	Example 2	
			Transmitter X.	Transmitter Y.
	L_a	$42°N$	$38°N$	$38°N$
	λ_a	$76°45'30''W$	$62°07'44''W$	$62°31'16''W$
	L_x	$20°13'48''S$	$43°58'08''N$	$35°15'17''N$
	λ_x	$36°45'30''W$	$68°07'44''W$	$75°31'16''W$
	$\lambda=\lvert\lambda_a-\lambda_x\rvert$	$40°$	$6°$	$13°$
	d	$72°22'29''$	$7°29'30''$	$10°46'20''$
	θ_0	$141°02'23''$	$35°27'46''$	$88°49'23''$
	θ_c	$140°44'23''$	$35°15'31''$	$100°47'22''$
	$\theta_0-\theta_c$	$18'00''$	$12'15''$	$-11°57'59''$
(4. 2)	$\log\tan\lambda$	9.92381	9.02162	9.36336
	$\log\sin L_a$	9.82551	9.78934	9.78934
	$\log\tan i$	9.74932	8.81096	9.15270
	i	$29°18'46''$	$3°42'$	$8°05'22''$
(4. 3)	$\log\cos\lambda$	9.88425	9.99761	9.98872
	$\log\cot L_a$	0.04556	0.10719	0.10719
	$\log\tan\psi$	9.92981	0.10480	0.09591
	ψ	$40°23'24''$	$51°50'48''$	$51°16'32''$
(4. 4)	$\log\cos\psi$	9.88176	9.79082	9.79628
	$\log\sec L_x$	0.02765	0.14284	0.08800
	$\log\sin d$	9.97912	9.11522	9.27162
	$\log\cot L_a$	0.04556	0.10719	0.10719
	$\log\frac{10,800}{\pi}$	3.53627	3.53627	3.53627
	$\log\tan(\theta_0-\theta_c)$	7.71900	7.55187	9.32623
	$\log r$	1.18936	0.24421	2.12559
	r	15.47	1.755	-133.53
(4. 1)	$\alpha=\theta_c+i$	$170°03'09''$	$38°57'$	$108°52'44''$

6. Analytic methods. In closing, two additional methods of accurately locating position by azimuths must be mentioned. The first is that in E. J. Willis' *Methods of Modern Navigation* [9] and obtains a fix without use of any map. To get analytically an azimuth line of position for the isoazimuth (X, θ_0), Willis writes the equations of: 1) the great circle through X cutting the parallel of dead reckoning latitude at the angle $90°-\theta_0$, and thus he finds the latitude and longitude of the point A in which that great circle cuts that parallel; 2) the great circle through X cutting the meridian of dead reckoning longitude at the angle θ_0, which gives the point B common to that meridian and the second great circle; 3) the great circle arc AB which, since A and B are on (X, θ_0), approxi-

mates (X, θ_0). The intersection of two great circle arcs obtained by step 3 from two stations can then be found analytically and provides a close approximation to the ship's position.

Another method which may be used with or without a map was suggested to me by Willis' analytic treatment of the Sumner line (*loc. cit.*). Differentiation of formula (2.2) of this paper yields:

$$-\csc^2\theta \sin\lambda d\theta = (\cot\phi_x \cos\phi + \cos\lambda \sin\phi)d\phi + (\sin\lambda\cos\phi - \cot\theta\cos\lambda)d\lambda. \tag{6.1}$$

If θ, ϕ and λ are computed for an assumed position and $d\theta = \theta_0 - \theta$, the equation in $d\phi$ and $d\lambda$ which results from (6.1) is that of a straight line whose $d\phi$ and $d\lambda$ intercepts are:

$$d\phi = \frac{-\csc^2\theta \sin\lambda d\theta}{\cot\phi_x \cos\phi + \cos\lambda \sin\phi} \tag{6.2}$$

$$d\lambda = \frac{\csc^2\theta \sin\lambda d\theta}{\cot\theta\cos\lambda - \sin\lambda\cos\phi}. \tag{6.3}$$

This straight line is in the tangent plane to the earth at the assumed position and the points $(d\phi, d\lambda)$ on the line represent to the first order of approximation those points $(\phi+d\phi, \lambda+d\lambda)$ of the earth on the isoazimuth (X, θ_0). This line may now be approximated by a line having the intercepts (6.2) and (6.3) on any map which represents distances near the assumed position with reasonable accuracy. It should be remarked, however, that plotting a line by its intercepts may require a much smaller scale than plotting by the method of sections 3–5 (see for instance example 2 in which one of the four intercepts is far off the map).

If the same assumed position is used for two isoazimuths, the two equations (6.1) may be solved simultaneously for $d\phi$ and $d\lambda$. Then $(\phi+d\phi, \lambda+d\lambda)$ is the "fix." Similarly if the same assumed position is used for an azimuth and a Sumner line, equation (6.1) and the corresponding equation in terms of differentials for the Sumner line, as given by Willis, may be solved simultaneously. These computations can be carried out without use of maps.

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6. H. Maurer, Annalen der Hydrographie, 1919, p. 16. See also EK p. 75 and fig. 16.

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8. If tables like those described in section 3 were available for whole degrees of L_a and λ and for half degrees of L_z, A should be chosen so that L_a and λ come out in whole degrees. The present H.O. 214 tables will serve to solve XAP using, in entering the tables, L_a for Lat., L_z for Dec. (wide interpolation may be necessary here since not all half degrees are tabulated) and λ for $H.A.$ Then if the names of L_a and L_z are the same, $\theta_c = Az$ (are opposite, $\theta_c = 180° - Az$) and $d = 90° - \text{alt.}$

9. D. Van Nostrand Co., 1928.

10. See also W. Immler, "Die Kartenbeschickung der Funkpeilung in der winkeltreuen Kegelkarte," Annalen der Hydrographie 68, 1940, 8, pp. 282–292; and P. de Vanssay de Blavous, "The position at sea by radiogoniometric bearings taken on board," Hydrographic Review 10, 1933, pp. 87–107.

WHAT IS A LATTICE?

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1. Order relations. With respect to one or more relations, most mathematical systems are lattices. Thus the real numbers are a lattice with respect to the *order* relation $x \leqq y$ (x is less than or equal to y); the non-negative integers are a lattice with respect to the relation $x \mid y$ (x is a *divisor* of y); the subsets of any class form a lattice if one relates them by the *inclusion* relation $X \subset Y$ (X is a subset of Y).

The assertion that the above three systems are *lattices* really means that the relations specified have a number of common formal properties. Thus the order relation between real numbers evidently has the following properties,

P1. For all x, $x \leqq x$, (Reflexivity)

P2. If $x \leqq y$ and $y \leqq x$, then $x = y$, (Anti-symmetry)

P3. If $x \leqq y$ and $y \leqq z$, then $x \leqq z$. (Transitivity)

Moreover statements P1–P3 remain true if the relation $\leqq$ of inequality between real numbers is replaced by the relation $\mid$ of divisibility between non-negative integers or by the relation $\subset$ of inclusion between subsets.

To summarize, the three relations $\leqq$, $\mid$, $\subset$, as defined, share properties P1–P3 *and all logical consequences of these properties*, however indirect.

2. A matter of economy. It seems to me a waste of paper to develop all properties of the inequality relation using one notation and terminology, to discuss divisibility using a second set of symbols and technical terms, and to repeat this procedure in treating the algebra of classes. Instead, I would take full ad-

vantage of basic analogies, and create a general *lattice theory* of relations satisfying P1–P3. This theory could contain as special cases many of the properties of inequality, divisibility, and inclusion; thus it would have advantages both of *unity* and *economy*.*

A trivial instance of this economy arises in deriving the properties of the relation $x<y$ and its analogs. By $x<y$, we mean that $x\leqq y$ but $x\neq y$. It is an easy but tiresome exercise to prove from P1–P3 that, for example, $x<y$ and $y\leqq z$ imply $x<z$; that $x_1<x_2<x_3<x_1$ is impossible, and so on. If lattice theory cannot entirely free us from making these tiresome proofs, it can at least free us from the need of going through them three or more times in different language. If we define a *proper divisor* of y to be an integer x such that $x\,|\,y$ yet $x\neq y$, we can apply *gratis* to this relation every property of $x<y$ proved from P1–P3. The same remark applies to the concept of a *proper subset.*

3. The principle of duality. A remarkable, but more sophisticated economy effected by lattice theory consists in its general Duality Principle, which includes as special cases duality principles in logic, projective geometry, number theory, *etc.* By the *dual* of a relation satisfying P1–P3, we mean simply its converse in the usual sense. Thus the dual of $\leqq$ is $\geqq$, the dual of $x\,|\,y$ is "x is a multiple of y," and the dual of $X\subset Y$ is $X\supset Y$.

In its simplest form, the Duality Principle states that the dual (converse) of any relation satisfying P1–P3 itself satisfies P1–P3. From this it follows that all definitions and theorems are either dual in pairs or self-dual.

For example, by a *lower bound* of a set X of elements X_i with respect to a relation $\leqq$ satisfying P1–P3, we mean simply an element u such that $u\leqq x_i$ for all x_i in X. The dual concept is that of *upper bound*, or element v such that $x_i\leqq v$ for all x_i in X. Analogs for divisibility are the dual concepts of common divisor and common multiple; what are the analogs for the inclusion relation between sets?

Similarly, by the *greatest lower bound* or *meet* of x, y with respect to $\leqq$, we mean an element u (denoted $x\cap y$) which is (i) a lower bound of x and y, and (ii) satisfies $u\geqq v$ for any other lower bound v of x and y. Thus in the case of inequality, $x\cap y$ is simply the *lesser* of x and y; in the case of divisibility, it is the *g.c.d.* of x and y; with inclusion, $X\cap Y$ is the *intersection* of X and Y, or set of all points in both X and Y.

The reader should have little trouble in defining the dual concept of the *least upper bound* or *join* of x and y, which is denoted $x\cup y$. This evidently specializes to the *greater* of x and y, the *l.c.m.* of x and y, and the *union* or sum of X and Y in the three examples we have taken.

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* Similar unifications, achieved by abstract group theory, field theory, and ideal theory, constitute the characteristic feature of modern algebra. The importance of economy in organizing scientific knowledge is described by the physicist and philosopher Ernst Mach in his "Science of Mechanics," Chicago, 1893, p. 6.

4. Lattices. We are now ready to define precisely what we mean when we speak of a lattice.

DEFINITION. *A lattice is a system L of elements $x, y, z, \cdots$ together with a relation which satisfies* P1–P3 *and also the postulate*

L. *Any two elements x and y have a meet $x \cap y$ and a join $x \cup y$ in L.*

THEOREM. *The following algebraic identities are true in any lattice:*

L1. *For all x, $x \cap x = x \cup x = x$,*

L2. *$x \cap y = y \cap x$ and $x \cup y = y \cup x$,*

L3. *$x \cap (y \cap z) = (x \cap y) \cap z$ and $x \cup (y \cup z) = (x \cup y) \cup z$.*

L4. *$x \cap (x \cup y) = x \cup (x \cap y) = x$.*

We omit the proof. Conversely, any system satisfying L1–L4 becomes a lattice if one defines $x \leqq y$ to mean $x \cup y = y$ (or equivalently, to mean $x = x \cap y$). It is largely because of this interpretation of lattices in terms of binary operations that lattice theory can be regarded as a branch of algebra.

In fact, there is an analogy between the operations $\cap$, $\cup$ in a lattice and the operations $\times$, $+$ of ordinary arithmetic. Like multiplication, the operation $\cap$ is commutative and associative; similarly for $\cup$ and addition (*cf.* L2–L3). Further, in the three examples cited, there is valid the analog

L6′. $x \cap (y \cup z) = (x \cup y) \cup (x \cap z)$

of the *distributive* law $x(y+z) = xy + xz$ of arithmetic.

However, the distributive law does not hold in all lattices. For example, take the points, lines, and planes of projective space, together with the void set O and space I. These are a lattice with respect to inclusion; $X \cap Y$ is the intersection of X and Y, while $X \cup Y$ is the linear sum of X and Y—that is, the set of all points on lines joining X and Y. It is easily checked that if x, y, z are three distinct points on a line L, then $y \cup z = L$ whence $x \cap (y \cup z) = x \cap L = x$, whereas $x \cap y = x \cap z = O$ and so $(x \cap y) \cup (x \cap z) = O$.

Indeed, it is curious and not easily proved fact, that if three elements in a lattice satisfy L6′, then they satisfy its dual,

L6″. $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$,

whose arithmetic analog $x + yz = (x+y)(x+z)$ is not true.

In the preceding example, the elements O and I are universal lower and upper bounds, respectively; that is, for all x, $O \leqq x \leqq I$. Such elements need not exist in all lattices (although they do in all finite lattices). Thus with real numbers we must adjoin the imaginary numbers $-\infty$ and $+\infty$ to get universal bounds; on the other hand, 1 is a common divisor and 0 a common multiple of all non-negative integers.

If universal bounds exist, it is easily shown that they are somewhat analogous to 0 and 1 in arithmetic. Thus we can show from P1–P3 and our definitions that, for all x,

$$O \cap x = O, \quad O \cup x = x, \quad \text{and} \quad I \cap x = x.$$

These are analogs of $0 \cdot x = 0$, $0 + x = x$, and $1 \cdot x = x$.

5. Applications in function theory and in logic. It would be possible to con-

tinue most indefinitely giving examples and properties of lattices. But I hope that the preceding examples will give some idea of the generality and simplicity of the notion of a lattice. In order to round out the picture, I shall conclude by mentioning two other lattices which play basic roles in function theory and logic.

The continuous real functions $f(x)$ of a real variable form a lattice, if one defines $f \leqq g$ to mean $f(x) \leqq g(x)$ for all x. Here $f \cap g$ is the function $h(x)$ which, for all x, is the lesser of $f(x)$ and $g(x)$; we define $f \cup g$ dually. Thus $f \cup -f$ is the *absolute* $|f(x)|$ of $f(x)$.

Finally, in mathematical logic, attributes (good, rich, female, *etc.*) form a lattice. Here $p \leqq q$ is interpreted to mean p is *implied* by q, $p \cap q$ to mean p *or* q, $p \cup q$ to mean p *and* q. In this lattice (often called a *Boolean algebra*), a fundamental role is also played by the operation p' (meaning *not* p). It satisfies

L7. $(p')' = p$, $p \cap p' = 0$, $p \cup p' = I$,

$$(p \cap q)' = p' \cup q', \quad \text{and} \quad (p \cup q)' = p' \cap q'.$$

It is noteworthy that the same laws are satisfied by sets, if one lets X' denote the *complement* of X (set of points not in X); further, they are satisfied in projective geometry if one lets x' denote the *polar* of x. In fact, it can be proved that the chief difference between projective geometry and Boolean algebra (or the algebra of logic) is that the distributive laws L6′–L6″ of Boolean algebra must be replaced in projective geometry by the weaker *modular* law.

L5. If $x \leqq z$, then $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$.

Since $x \leqq z$, clearly $x \cup z = z$, so that the conclusion of L5 assumes the *self-dual* form $x \cup (y \cap z) = (x \cup y) \cap z$; thus it is also equivalent to $(x \cup y) \cap z = (x \cap z) \cup (y \cap z)$.

The self-dual modular law L5 is important for another reason. It is satisfied by the normal subgroups of any group, the ideals of any ring, *etc.*, and may be made the basis of most of the known decomposition theorems of modern algebra. But this is a long story!

AN EARLY REFERENCE TO DIVISION BY ZERO

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1. The Hindu legend. In this MONTHLY for 1924 we read* that Brahmagupta, writing in 628 A.D., was the first person to speak of division by cipher, although he gave no quotient. We read further that Bhaskara in 1152 termed such a quotient infinite. This is a reiteration of statements which one can find in almost any history or reference work in mathematics.† It is but one aspect of the impression generally prevailing that "the arithmetic of zero is entirely the Hindu contribution to the development of mathematical science. With no

* H. G. Romig, "Early history of division by zero," this MONTHLY, 31, 1924, 387–389.

† See Encyclopédie des sciences mathématiques, I, 1^1, 1904, p. 33; Moritz Cantor, Vorlesungen über Geschichte der Mathematik, I, 2nd ed., Leipzig, 1894, p. 576; H. T. Colebrooke, Algebra, with Arithmetic and Mensuration, from the Sanscrit of Brahmagupta and Bhaskara, London, 1817, pp. 137–138, 339–340; B. Datta and A. N. Singh, History of Hindu Mathematics. A Source Book. Part I. Numeral Notation and Arithmetic, Lahore, 1935, pp. 238–243.

other early nations do we find any treatment of zero."* Such categorical assertions are serious misrepresentations of the true situation.

It is now well known that symbols for empty places in the positional representation of numbers—corresponding in this respect to our zero—were used by the Babylonians, the Greeks, and the Mayas. Such use antedates historical evidence for the Hindu zero by some thousand years. Nevertheless, such symbols did not appear alone, or independently of other ciphers or digits, to represent zero as itself a number; they were simply part of the representations of positive integers and fractions. Hence the adoption of such symbolisms does not necessarily refute the Hindu claims of priority to the use of zero in this independent sense. Nevertheless, the numerational concept of zero—as distinct from a symbol for empty places—was also familiar to the ancient Greeks. This idea frequently is attributed to Plato,† but such ascription rests largely on gratuitous inferences derived from his sybilline phraseology. In the works of his pupil Aristotle, however, there appears a clear-cut reference not only to the numerical notion of zero but also to the result of division by zero. Historians of mathematics seem generally to have overlooked‡ this significant passage in the *Physics* which anticipates the earliest extant Hindu evidence in this connection by almost a millenium.

2. Aristotle's law of motion. A study of moving bodies had led Aristotle to conclude§ that for a given force or impulse the velocity was in all cases inversely

* Bibhutibuhusan Datta, "Early history of the arithmetic of zero and infinity in India," Bulletin of the Calcutta Mathematical Society, XVIII, 1927, 165–176. See articles by the same author in this MONTHLY for 1926 and 1931.

† See John Burnet, Greek Philosophy. Thales to Plato, London, 1932, pp. 320–322, 330; A. E. Taylor, Plato. The Man and his Work, new ed., New York, 1936, pp. 505–506.

‡ The passage is noted quite incidentally in a philosophical treatment by Albert Görland, Aristoteles und die Arithmetik, Marburg, 1898, p. 18, with no indication of its mathematical significance. There are numerous other works on Aristotle's general mathematical thought, and more particularly on his concept of number and his notion of infinity; but these appear to contain no reference to his statement on division by zero. See, for example, Max Dehn, "Raum, Zeit, Zahl bei Aristoteles, vom mathematischen Standpunkt aus," Scientia, LX, 1936, 12–21, 69–74, supp. pp. 1–9, 32–36; Abel Burja, "Sur les connoissances mathématiques d'Aristote," Preussische Akademie der Wissenschaften, Berlin, Mémoires de l'académie royale des sciences et belles lettres, 1790–1791, pp. 257–276; J. L. Heiberg, "Mathematisches zu Aristoteles," Abhandlungen zur Geschichte der mathematischen Wissenschaften, 18, 1904, 1–49; Gaston Milhaud, "Aristote et les mathématiques," Archiv für Geschichte der Philosophie, 16, 1903, 367–392; Paul Mansion, Aristote et les mathématiques," Revue de Philosophie, 3, 1903, 832–834; Abraham Edel, Aristotle's Theory of the Infinite, New York, 1934. I have not had an opportunity to examine the following: Josephus Blancanus, Aristotelis Loca Mathematica ex Universis Ipsius Operibus Collecta, et Explicata, Bononiae, 1615; P. J. Monzó, De Locis apud Aristotelem Mathematicis, Valencia, 1556; Leo Reiche, Das Problem des Unendlichen bei Aristoteles, Breslau, 1911; Julius Stenzel, Zahl und Gestalt bei Plato und Aristoteles, Leipzig, 1924; R. Stölze, Die Lehre vom Unendlichen bei Aristoteles, Würzburg, 1882. The excellent source book by Ivor Thomas, Selections Illustrating the History of Greek Mathematics, 2 vols., Cambridge, Mass., 1939–1941, devotes more than a dozen pages to Aristotle, but does not include this material. So far as I am aware, there is no reference to this point in any history or textbook of mathematics.

§ That this notorious inference is incorrect need not concern us here; but it should perhaps be noted that only in the time of Newton was a more accurate study made of the difficult problem

proportional to the density of the medium—such as air or water—through which the object moved. In modern symbolism this principle may be represented in the form of an equation, $v = k/\delta$, where v, the speed, is a function of δ, the density of the medium, and k is a factor of proportionality. Aristotle then raised the question as to what would be the speed of a body in a vacuum—that is, when $\delta = 0$. In answer he wrote:

> Now there is no ratio in which the void is exceeded by body, as there is no ratio of 0 to a number. For if 4 exceeds 3 by 1, and 2 by more than 1, and 1 by still more than it exceeds 2, still there is no ratio by which it exceeds 0; for that which exceeds must be divisible into the excess + that which is exceeded, so that 4 will be what it exceeds 0 by +0. For this reason, too, a line does not exceed a point—unless it is composed of points!*

3. Impossibility of division by zero. In this quotation it is quite evident that Aristotle had the arithmetical zero in mind. It is clearly distinguished from the philosophical void, and it is regarded as akin to, although not actually one of, the numbers. Zero is looked upon as bearing to number the same relationship as does a point to a line. Moreover, the impossibility of division by zero is here explicitly stated almost fifteen hundred years before the time of Bhaskara. The argument by which Aristotle excluded division by zero is based largely on the traditional meanings of words, and hence it differs from the modern point of view. But there can be no doubt as to his correct understanding of the situation. If division by zero were possible, then the result would exceed every integer. That this is Aristotle's position is clear from his inference with respect to motion in a vacuum:

> But if a thing moves through the thickest medium such and such a distance in such and such a time, it moves through the void with a speed beyond any ratio.†

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of the influence of a resisting medium. Indeed, the assumption of Aristotle was applied to the velocity of light both by Descartes in illustrating the law of refraction and by Fermat in demonstrating that the principle of "least time" satisfies this law.

* Physica IV. 8. 215b. The translation is taken from The Works of Aristotle, ed. by W. D. Ross and J. A. Smith, 11 vols., Oxford, 1908–1931, vol. II. The same translation is found in The Basic Works of Aristotle, ed. by Richard McKeon, 1941. This rendering of the Greek is misleading, however, in the use of symbols for numbers and operations. Aristotle did not here use the Greek symbol *o* for zero, but rather the word *οὐδεν*, corresponding to the Hindu *sunya*, "void," and the Arabic *sifra* (whence our *cipher*). In this respect the translation given by P. H. Wicksteed and F. M. Cornford, Aristotle, The Physics, 2 vols., London, 1929–1934 is somewhat more cautious, if also more diffuse: "But the nonexistent substantiality of vacuity cannot bear any ratio whatever to the substantiality of any material substance, any more than zero can bear a ratio to a number. For if we divide a constant quantity c (that which exceeds) into two variable parts, a (the excess) and b (the exceeded), then, as a increases, b will decrease and the ratio $a:b$ will increase; but when the whole of c is in section a there will be none of c for section b; and it is absurd to speak of 'none of c' as 'a part of c.' So the ratio $a:b$ will cease to exist, because b has ceased to exist and only a is left, and there is no proportion between something and nothing. (And in the same way there is no such thing as the proportion between a line and a point, because, since a point is no part of a line, taking a point is not taking any of the line.)"

† *Ibid.* The translation is again that of Ross and Smith. Wicksteed and Cornford render it:

He easily establishes this fact through a reductio ad absurdum: If there were a a ratio, it would follow that a body in a given time would traverse a certain distance whatever the density of the medium, a result which contradicts the principle of the premise,* $v = k/\delta$.

4. Aristotle vs. Bhaskara. It is illuminating to compare these passages from Aristotle with the much vaunted statement on the subject of division by zero given by Bhaskara:

> Statement: Dividend 3. Divisor 0. Quotient the fraction 3/0.
>
> This fraction, of which the denominator is cipher, is termed an infinite quantity.
>
> In this quantity consisting of that which has cipher for its divisor, there is no alteration, though many be inserted or extracted; as no change takes place in the infinite and immutable God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth.†

A comparison of the above quotations from Aristotle and Bhaskara will show that the view of the former comes closer to the modern attitude than does the much more recent dictum of the latter. In contemporary mathematics it is indeed customary to say that the quotient of any number (other than zero) divided by zero is infinity. This language, however, is conventionally interpreted as signifying two things: first, that in this case the quotient is not defined and hence the indicated division is impossible; second, that as a variable divisor tends indefinitely closely toward zero, the (non-zero) dividend remaining constant, the quotient increases without limit. Both of these aspects are present in Aristotle's statement, the former categorically stated and the latter clearly implied.‡ Bhaskara, on the other hand, did not assert the impossibility of division by zero; nor is there in his form of expression any indication of the indefinite increase of $3 \div x$ as x tends towards 0. He looked upon $3 \div 0$ as a number having unusual properties, but very definitely a fixed quantity, much as Wallis in 1665 boldly but uncritically wrote $1/0 = \infty$. Bhaskara's further remark that 3/0 remains unchanged, whatever be added to or taken from it, is sometimes interpreted as a significant anticipation of the properties of the modern infinite; but it must be remarked that the contemporary viewpoint of infinity is con-

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"But if movement through the finest medium covers a given distance in a given time, movement through the void is out of all proportion."

* It was this obvious contradiction which led Aristotle to deny the existence of the void. This fact indicates that the Peripatetic doctrine, "Nature abhors a vacuum," was not based on animistic or even teleological notions, but was instead a logical consequence of a physical principle which science ultimately found cause to reject.

† Colebrooke, *op. cit.*, pp. 137–138.

‡ The implication in this connection is more clearly expressed in the Wicksteed-Cornford translation than in that of Ross and Smith. See note above.

cerned with infinite sets and one-to-one correspondence, or with improper geometrical elements, rather than with magnitudes and equality in the ordinary sense. Hence Bhaskara's naïve assumption (typical of Hindu arithmetic) that division by zero results in a quasi-theological ineffable fixed infinity is less to be admired than Aristotle's critical recognition (characteristic of Greek thought) of the impossibility of division by zero.*

5. Judgment suspended. That Aristotle was the first person to discuss this question is not unlikely, but this cannot be held as established. One may perhaps be permitted to conjecture from the very casualness of Aristotle's language in this connection that the arithmetic of zero was more or less well known in his day. However, there seems to be no other extant reference to division by zero before the time of Brahmagupta. As a matter of fact, the Greeks, Aristotle included, did not look upon zero as a number in the strict sense of the word, for this status was reserved exclusively for the natural numbers.† Statements on the arithmetic of zero are rare in ancient times,‡ and indeed zero was not fully accepted in algebra until the sixteenth and seventeenth centuries.§ Aristotle himself seems never to have had occasion to consider the quotient of zero divided by zero. The earliest, but quite inadequate, consideration of this problem traditionally has been ascribed to Brahmagupta:

> Positive, divided by positive, or negative by negative, is affirmative. Cipher, divided by cipher, is nought. Positive, divided by negative, is negative. Negative, divided by affirmative, is negative. Positive, or negative, divided by cipher, is a fraction with that for denominator: or cipher divided by negative or affirmative.‖

Tradition in this particular respect may prove to be trustworthy, but it necessarily must be rejected with respect to the more general problem. Historical evidence points to Aristotle, rather than to Brahmagupta, as the one who first considered division by zero.

* Incidentally, one finds in Aristotle's Physica a discussion of the infinite which, while differing markedly from the modern treatment, shows an admirable grasp of the problems.

† The integers one and two frequently were excluded, together with zero, from the realm of number. This situation, however, does not alter the argument of this paper. That the Greeks had a thorough knowledge of the arithmetic of ratios and of the integers one and two—even though these were not strictly defined as numbers—indicates that a corresponding arithmetic of zero was quite possible, notwithstanding the nice distinctions which placed zero beyond the scope of the word number.

‡ Nicomachus (c. 100 A.D.) in his Arithmetica referred only once, and then somewhat equivocally, to zero in saying that the sum of nothing added to nothing is nothing. See Nicomachus of Gerasa, Introduction to Arithmetic, transl. by M. L. D'Ooge, with Studies in Greek Arithmetic by F. E. Robbins and L. C. Karpinski, New York, 1926, pp. 48, 120, 237–238. Cf. Johannes Tropfke, Geschichte der Elementar-Mathematik, Vol. II, Leipzig, 1921, pp. 56–59.

§ See Tropfke, *loc. cit. Cf.* Encyclopédie des sciences mathématiques, I, 1^1, p. 133, note 147; also G. Eneström, "Über die Anfänge der Benutzung von Null als eine wirkliche Grösse," Bibliotheca Mathematica (3), VII, 1906–1907, 309.

‖ Colebrooke, *op. cit.*, pp. 339–340.

THE GEOMETRIC FOUNDATIONS OF THE MECHANICS OF A RIGID BODY*

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1. Introduction. The concepts of geometry, even the simplest ones such as the point, the straight line, the plane, have been created by an idealization of physical experience. The geometric system has then been developed by means of a logical organization of this material. Thus the mechanics of a rigid body has given rise to some interesting chapters of geometry, that is, to investigations which are based on the axioms and are amenable to the methods of geometry.

Speaking of vectors, tensors and dynames Felix Klein says in one of his papers: "We have thus obtained through systematic, geometric investigation all those quantities which the physicist must always have at hand. It is *pure geometry*, however, that we are studying. I must emphasize this all the more since these things are often regarded as belonging to physics and are therefore discussed in books and lectures on physics instead of on geometry. In the nature of the case such an attitude is thoroughly unjustified. When the time came physics had to create the weapons which it needed and which it did not find ready in mathematics" [5].

Let us consider these peaceful weapons more closely. In order to introduce the concept of the so called *dyname* which is our main subject we start with some well known facts. A single force acting on a rigid body is not a vector as in the mechanics of the particle but a so called "line bound" vector or "gliding" vector which can only be moved *along its line of action*, which is also called its *axis*. This five-dimensional magnitude has been called a *bar*, in order to distinguish it clearly from a vector, and we may define it by saying that: 1) Each directed couple of points defines a bar; 2) all directed couples of points which can be transformed into each other by translation along one and the same straight line define the same bar (Fig. 1); whereas for a vector any arbitrary parallel translation of the directed couple is admissible.

Each bar $\mathfrak{A}$ (Fig. 2) defines a vector $\mathbf{a}$ and a moment vector field

$$\mathbf{a}_o = \mathbf{r} \times \mathbf{a}, \tag{1}$$

where $\mathbf{r} \times \mathbf{a}$ is the vector product of these two vectors. The vector $\mathbf{a}_o$ is called the moment of the force about the point o. For any other point p it follows from the figure that

$$\mathbf{a}_p = \mathbf{a}_o + (po \times \mathbf{a}). \tag{2}$$

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* Lecture delivered by invitation in connection with the program of "Advanced Instruction and Research in Mechanics," Providence, R. I., July 11, 1942.

We see from (2) that the entire moment field is determined by $\mathbf{a}$ and the moment about *any one* particular point, *e.g.* the point o. Hence we may take the three components of $\mathbf{a}$ and the three components of $\mathbf{a}_o$ as the coördinates of the bar. But, because of (1) there exists between these six numbers the identical relation

$$I \equiv \mathbf{a} \cdot \mathbf{a}_o = 0 \qquad \text{(Plücker's relation).} \tag{3}$$

If instead of $\mathbf{a} \cdot \mathbf{a}_o$ we consider the scalar product of $\mathbf{a}$ and $\mathbf{a}_p$ (Fig. 2) from (2) we see also that $\mathbf{a} \cdot \mathbf{a}_p = 0$.

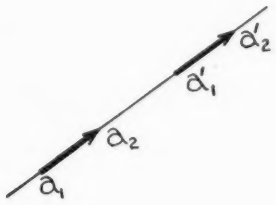

FIG. 1

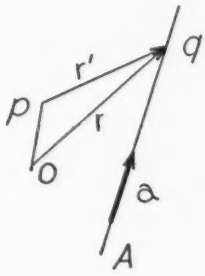

FIG. 2

If we define *addition of two bars* $\mathfrak{A}$ and $\mathfrak{B}$ as the addition of their resultant vectors and their moment vectors:

$$\mathfrak{C} = \mathfrak{A} + \mathfrak{B} \quad \text{if} \quad \mathbf{c} = \mathbf{a} + \mathbf{b}; \qquad \mathbf{c}_o = \mathbf{a}_o + \mathbf{b}_o, \tag{4}$$

we see that the "invariant" $\mathbf{c} \cdot \mathbf{c}_o$ of the bar $\mathfrak{C}$ may now be $\neq 0$, *i.e. the sum of two bars is in general no longer a bar but a new six-dimensional magnitude.* These magnitudes have been called *screws*, *dynames* or *motors*.

May I say a few words regarding the history of our subject. The most important earlier contributions to the theory of screws are due to Poinsot [8] who investigated the existence of the so called *central axis*, to Chasles [2] and to Plücker [7] who emphasized the point of view of line geometry and introduced the word "dyname." In 1900 appeared the great work of Sir Robert Ball: "A Treatise on the Theory of Screws," [1] where he presented the subject to the English-speaking world and developed systematically the theory and applications of screws. This work aroused the interest of Felix Klein who in a paper [4] "On the Theory of Screws of Sir Robert Ball" discussed its main ideas from the point of view of the theory of groups of transformations.

In 1903 there appeared E. Study's important book "The Geometry of Dynames" [9]. I should like to mention just two of his ideas which I am going to discuss in detail presently. The first is the definition of the dyname or the *motor*,

as Study calls it, as a "directed couple of straight lines," in analogy with the definition of a vector as a directed couple of points. The second idea I want to indicate is his *principle of correspondence*, which Blaschke, a great admirer of Study, considers as the most important content of Study's great work. It is based on the use of a new kind of complex numbers, "dual numbers," originally due to Clifford. In 1924 v. Mises [6] developed a motor calculus designed to supplement the vector calculus in certain important domains of mechanics. He defines a *scalar product* and a *motor product* of two motors which have elementary mechanical meanings. Another important step is the introduction of *motors of the second order*, or motor tensors.

2. The dyname of forces and the dyname of velocity. Let us give more details along the lines just indicated. If we add n bars

$$\mathfrak{F} = \sum_{i=1}^{n} \mathfrak{A}_i \tag{4'}$$

instead of two we do not get anything new. The resultant vector and the moment vector of the sum will be given by

$$\mathbf{F} = \sum_{i=1}^{n} \mathbf{a}_i, \qquad \mathbf{M}_0 = \sum_{i=1}^{n} (\mathbf{r}_i \times \mathbf{a}_i), \tag{4''}$$

where $\mathbf{r}_i$ is the vector from the point o to any point on the line of action of the ith bar. For any other point p we have, as in (2),

$$\mathbf{M}_p = \mathbf{M}_o + (po \times \mathbf{F}) \tag{5}$$

Thus the original system of forces is so far equivalent *to a single force whose line passes through* o and whose vector is equal to the vector sum $\mathbf{F}$ of the n forces *and to a couple* whose torque $\mathbf{M}_o$ is equal to the sum of the torques about o of the n separate forces. Scalar multiplication of (5) by $\mathbf{F}$ gives

$$\mathbf{M}_p \cdot \mathbf{F} = \mathbf{M}_o \cdot \mathbf{F} \equiv I \tag{6}$$

If this invariant $I=0$, $\mathfrak{F}$ is *a single bar*, or a *couple*. If $I \neq 0$, $\mathfrak{F}$ *is a general dyname*. The sum of several dynames is again a dyname.

In the case of the bar we see from (2) that the moment vector of any point on the axis of the bar is zero. In the general case, where $I \neq 0$, Poinsot has shown that *there exists one and only one straight line having the direction of* $\mathbf{F}$ *and such that the moment vector* $\mathbf{M}_o$ *about any point of this line is parallel to the resultant vector* $\mathbf{F}$. *The moment vector is the same for all the points on this line, and its magnitude is smaller than for any other point not on the line.* This line is called the *axis of the dyname* or Poinsot's *central axis* and we may now give a dyname by its central axis and by the two scalar magnitudes $|\mathbf{F}|$ and $|\mathbf{M}_o|$. Interpreting our results in terms of forces we see that *any system of forces acting on a rigid body is equivalent to a dyname.*

There is a well known duality between the most general *system of forces* in space and the most general *state of motion* of a rigid body in space. Both are represented by dynames. While the fundamental axioms of statics are axioms of equilibrium, kinematics is based on the axioms of geometry. The following geometric theorem due to Chasles is basic in the study of kinematics in space: *If a congruent transformation possesses one self-corresponding point it possesses a whole straight line of such points which is called the axis of the transformation.* Or in the language of physics: *Any displacement of a rigid body with one fixed point o is equivalent to a rotation about a definite axis through o.* Next we consider the general motion in which no point is fixed. Such a motion is equivalent to *a translation plus a rotation about an axis.* For consider any point o and translate the body so as to bring o into its final position. Then the remaining motion has this point fixed and hence, by the theorem of Chasles, consists of a rotation about an axis through this point. So far these are *finite* displacements; the concept of "velocity" has not entered into our considerations. In the case of a general *continuous* motion we can establish, assuming merely the rigidity of the body, the basic equation

$$\mathbf{v} = \mathbf{v}_o + (\omega \times \mathbf{r}) \tag{7}$$

where $\mathbf{v}$ and $\mathbf{v}_o$ are the velocities of points p and o respectively, $\mathbf{r} = op$ and ω is a constant vector. Equation (7) is known as *Euler's equation.* It has the same form as equation (5) for the transformation of the moment of a dyname. Therefore let us introduce the "velocity dyname" $\mathfrak{V}$ with "resultant vector" ω and definite axis parallel to ω, the vector of *angular velocity*; the "moment vector" $\mathbf{v}_o$ about any point of the axis is called the vector of *translation.* Thus the most general velocity is given at each instant by a dyname and the motion itself is a screw motion about an instantaneous axis.

Instead of giving a dyname $\mathfrak{F}$ by its axis and two scalar numbers we may represent it by the resultant vector $\mathbf{F}$ and moment vector $\mathbf{M}_p$ where p is such that $\mathbf{M}_p$ is not parallel to $\mathbf{F}$. In the correspondence between the dyname of forces and the dyname of velocity

$$\text{(I)} \quad \mathfrak{F}\text{: } (\mathbf{F}, \mathbf{M}_p) \quad \text{and} \quad \text{(II)} \quad \mathfrak{V}\text{: } (\omega, \mathbf{v}_p)$$

the resultant vector $\mathbf{F}$ corresponds to the angular velocity ω and the moment vector $\mathbf{M}_p$ to the velocity $\mathbf{v}_p$. We then have the following cases: If $\mathbf{F}=0$, $\mathbf{M}_p \neq 0$ we have in (I) a *couple of forces*, in the corresponding case (II) a *pure translation*; if $\mathbf{F} \neq 0$ but $I=0$ we have a bar, *i.e.* a *single force* in (I), a *pure rotation* in (II); and if $I \neq 0$ we have in both cases the *general dyname.*

The axis of the dyname may be considered as the axis of a *null-system.* For every point p in space the normal to the corresponding null plane π defines the direction of the moment vector $\mathbf{M}_p$ (or of the velocity $\mathbf{v}_p$). In this sense F. Klein calls the *linear complex the representant of the dyname.*

3. Motor geometry and motor calculus. An important step forward was taken by E. Study when he defined a motor as a *directed couple of straight lines.* Imagine two straight lines which are neither parallel nor perpendicular. Then there exists one and only one common normal which we call their *axis*, and we define the motor by the following specifications: 1) A directed couple of straight lines A_1, A_2 defines a "motor"; 2) all directed couples of straight lines which can be transformed into each other by a screw motion along their axis define the same motor (Fig. 3). The motor is obviously completely independent of any coordinate system.

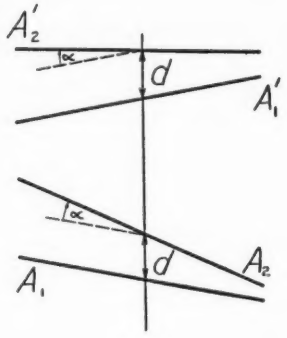

FIG. 3

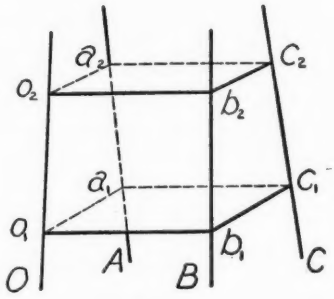

FIG. 4

If we interpret the motor as a state of motion we may regard $\tan\alpha$ as a measure of the angular velocity and the distance as a measure of the translation. In fact we can put:

$$d = \operatorname{dist}(A_1, A_2) = |\mathbf{A}_o|, \qquad \tan\alpha = |\mathbf{A}|, \tag{8}$$

where $\mathbf{A}_o$ is to be considered as the moment vector with respect to a point o on the axis, independent of any interpretation.

From these definitions Study deduces an extremely simple construction of the moment vector $\mathbf{A}_p$ for any point p: Suppose, without loss of generality, that p is on the straight line A_1: Denote *by q the point of intersection of A_2 with a plane Σ through p, perpendicular to A_1. Then we see easily that $pq = \mathbf{A}_p$.* On this result he bases his remarkable construction of the *sum of two motors* $\mathfrak{A}$ *and* $\mathfrak{B}$:

We choose as the common "first line" of both $\mathfrak{A}$ and $\mathfrak{B}$ the common perpendicular O of their axes. Let the "second lines" be A and B respectively (Fig. 4). In any plane Σ_1 perpendicular to O we construct the parallelogram o_1, a_1, b_1, c_1 where o_1, a_1, b_1 are the points of intersection of Σ_1 with O, A, B respectively. We do the same with another plane Σ_2 perpendicular to O, obtaining the parallelogram o_2, a_2, b_2, c_2. Then the locus of the points c is the straight line $c_1c_2 = C$, and the lines O and C determine the motor $\mathfrak{C} = \mathfrak{A} + \mathfrak{B}$. In fact it follows from the foregoing construction that

$$o_1a_1 = \mathbf{A}_{o_1}, \qquad o_1b_1 = \mathbf{B}_{o_1}, \qquad o_1c_1 = \mathbf{C}_{o_1},$$

and similarly for a_2, b_2, c_2. Hence

$$\mathbf{A}_{o_1} + \mathbf{B}_{o_1} = \mathbf{C}_{o_1}, \qquad \mathbf{A}_{o_2} + \mathbf{B}_{o_2} = \mathbf{C}_{o_2}; \tag{9'}$$

and since this is true for *two* points we can conclude that

$$\mathbf{A} + \mathbf{B} = \mathbf{C}. \tag{9''}$$

We see from (9′) and (9″) that this simple construction is equivalent to the *analytic addition of motors by the addition of their "components."* In fact, if we denote by A_1, A_2, A_3 and by A_4, A_5, A_6 the components of $\mathbf{A}$ and $\mathbf{A}_p$, respectively and similarly for $\mathbf{B}$ and $\mathbf{B}_p$ and $\mathbf{C}$ and $\mathbf{C}_p$ (where $\mathbf{A}_p$ is not parallel to $\mathbf{A}$ and similarly for $\mathbf{B}_p$ and $\mathbf{C}_p$) it follows from (9′) and (9″) that the sum $\mathfrak{C}=\mathfrak{A}+\mathfrak{B}$ determined by Study's construction corresponds to

$$C_i = A_i + B_i \qquad (i = 1, 2, \cdots, 6). \tag{9'''}$$

Next we define *multiplication.* The *scalar product* $\mathfrak{A}\cdot\mathfrak{B}$ of two motors $\mathfrak{A}$ and $\mathfrak{B}$ is

$$\mathfrak{A}\cdot\mathfrak{B} = \mathbf{A}\cdot\mathbf{B}_o + \mathbf{A}_o\cdot\mathbf{B} = A_1B_4 + A_2B_5 + A_3B_6 + \cdots + A_6B_3. \tag{10}$$

This definition is *independent of the coordinate system,* as may easily be seen. As an application consider the motion of a rigid body under a system of forces. Let this system be represented by the motor $\mathfrak{F}$:

$$\mathbf{F}: \ (F_x, F_y, F_z); \qquad \mathbf{M}_p: \ (M_x, M_y, M_z),$$

and let the velocity motor be represented by $\mathfrak{V}$:

$$\boldsymbol{\omega}: \ (\omega_x, \omega_y, \omega_z); \qquad \mathbf{v}_p: \ (v_x, v_y, v_z),$$

Then the scalar product

$$\mathfrak{F}\cdot\mathfrak{V} = F_xv_x + F_yv_y + F_zv_z + M_x\omega_x + M_y\omega_y + M_z\omega_z \tag{10'}$$

is the *power absorbed by the body.*

The motor product

$$\mathfrak{P} = \mathfrak{A}\times\mathfrak{B} \tag{11}$$

means

$$\mathbf{P} = \mathbf{A}\times\mathbf{B}, \qquad \mathbf{P}_p = (\mathbf{A}\times\mathbf{B}_p) + (\mathbf{A}_p\times\mathbf{B}). \tag{11'}$$

Here $\mathbf{P}$ is clearly independent of the point p, and for $\mathbf{P}_p$ we find by an easy computation that

$$\mathbf{P}_{p'} = \mathbf{P}_p + (\mathbf{r}\times\mathbf{P}), \quad \text{where} \quad \mathbf{r} = pp', \tag{12}$$

as it ought to be for a moment vector according to (5). Hence $\mathfrak{P}$ is indeed a motor. One important application of this product is to the analogue of the so called *Euler rule of differentiation*: Consider a coordinate system revolving with

angular velocity ω. We then have by Euler's rule for any vector $\mathbf{a}$

$$\frac{d\mathbf{a}}{dt} = \frac{d'\mathbf{a}}{dt} + (\boldsymbol{\omega} \times \mathbf{a}), \tag{13}$$

where d'/dt is the rate of change with respect to the rotating system and d/dt the rate of change with respect to a system at rest. More generally, with the same differential notation, let the motion of the coordinate system be given by a velocity motor $\mathfrak{V}$. Then it can be shown that

$$\frac{d\mathfrak{A}}{dt} = \frac{d'\mathfrak{A}}{dt} + (\mathfrak{V} \times \mathfrak{A}). \tag{14}$$

We shall use (14) in a problem belonging to the theory of flight (§5).

It is possible to use the motor calculus in line geometry as the vector calculus is used in point geometry. But for these purely geometric applications it seems that Study's *dual numbers* are perhaps more appropriate. Let me briefly explain Study's idea. Let the components of $\mathbf{a}$ and $\mathbf{a}_o = \mathbf{r} \times \mathbf{a}$ where o is the origin and $\mathbf{r}$ a vector from o to any point of the straight line be the Plücker coordinates of a straight line. These are six numbers, but they reduce to four because

$$\mathbf{a}^2 = 1 \quad \text{and} \quad \mathbf{a} \cdot \mathbf{a}_o = 0. \tag{15}$$

We now introduce with Study the *dual vector*

$$\mathbf{A} = \mathbf{a} + \epsilon \mathbf{a}_o \quad \text{where} \quad \epsilon^2 = 0. \tag{16}$$

Then from (16)

$$\mathbf{A}^2 = (\mathbf{a} + \epsilon \mathbf{a}_o)^2 = \mathbf{a}^2 + 2\epsilon \mathbf{a} \cdot \mathbf{a}_o = 1. \tag{17}$$

Hence (16) defines a one to one correspondence *between the dual points on the unit sphere and the four dimensional space of line geometry*. This is Study's *principle of correspondence*, which enables him to build up a new analytic geometry of straight lines.

If we have *two* dual vectors $\mathbf{A}$ and $\mathbf{B}$ we may form their product

$$\begin{aligned} \mathbf{AB} &= (\mathbf{a} + \epsilon \mathbf{a}_o) \cdot (\mathbf{b} + \epsilon \mathbf{b}_o) \\ &= \mathbf{a} \cdot \mathbf{b} + \epsilon (\mathbf{a}_o \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}_o). \end{aligned} \tag{18}$$

Here $\mathbf{a} \cdot \mathbf{b} = \cos \phi$, where ϕ is the angle of the two lines, and

$$\mathbf{a}_o \cdot \mathbf{b} + \mathbf{b}_o \cdot \mathbf{a} = -d \sin \phi, \tag{19}$$

where d is their shortest distance. These magnitudes are clearly invariant under a screw motion about the axis of the two lines. The *dual angle* of two straight lines has been defined:

$$\Phi = \phi + \epsilon d; \tag{20}$$

it follows that

$$\cos \Phi = \cos \phi \cos \epsilon d - \sin \phi \sin \epsilon d = \cos \phi - \epsilon d \sin \phi, \tag{21}$$

because for any angle α, $\cos \epsilon\alpha = 1$, $\sin \epsilon\alpha = \epsilon\alpha$, according to the definition of sine and cosine by their power series and the fact that $\epsilon^2 = 0$. Hence

$$\mathbf{AB} = \cos \Phi, \tag{22}$$

i.e. $\mathbf{AB}$ *is equal to the dual cosine* of the two lines. The equation $\mathbf{AB} = 0$ is the condition that the two lines *intersect at right angles.*

If we now consider $\mathbf{A}$ *as a function of a scalar parameter* t:

$$\mathbf{A}(t) = \mathbf{a}(t) + \epsilon\mathbf{a}_o(t), \tag{23}$$

this equation defines a *ruled surface*; and a system of straight lines dependent on *two* parameters:

$$\mathbf{A}(u, v) = \mathbf{a}(u, v) + \epsilon\mathbf{a}_o(u, v) \tag{24}$$

defines a *line congruence.*

If we do *not* assume $\mathbf{a}^2 = 1$ and $\mathbf{a} \cdot \mathbf{a}_o = 0$ the dual vector is a six-dimensional magnitude. Then $\mathbf{a}_o \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}_o$ is the same as in (10). It seems that the dual numbers are very useful for purposes of line geometry but that motor calculus has definite advantages for problems in mechanics.

4. Motor dyadics and dynamical equations of a rigid body. We may define a motor tensor or *motor dyadic* as a *transformation* Π which associates with every possible motor $\mathfrak{M}$ a transformed motor $\mathfrak{M}'$ by a law which is *linear* with respect to the components of $\mathfrak{M}$:

$$\mathfrak{M}' = \Pi \cdot \mathfrak{M} = M_1\mathfrak{P}_4 + M_2\mathfrak{P}_5 + \cdots + M_6\mathfrak{P}_3. \tag{25}$$

The six equations corresponding to (25) are

$$\begin{aligned} M_1' &= P_{11}M_4 + \cdots + P_{61}M_3 \\ &\cdots\cdots\cdots\cdots\cdots \\ M_6' &= P_{16}M_4 + \cdots + P_{66}M_3. \end{aligned} \tag{26}$$

The thirty-six scalar numbers $P_{11}, \cdots, P_{66}$ constitute the "components" of this tensor.

One of the most important motor dyadics is the *dyadic of inertia*. The inertia of a *particle* is described by *one* scalar number, its *mass*; the *inertia of a rigid body* requires more data: the *mass* m, the position $\mathbf{r}$ of the *centroid*, the *moments* and the *products of inertia*. It can be shown that all these magnitudes are elements of a single rather particular motor dyadic which thus describes all inertia properties of the body. If we introduce

$$\int dm = m, \quad \int x dm = ma, \cdots, \quad \int (x^2 + y^2) dm = I_z, \cdots,$$

$$\int xy dm = D_z, \cdots,$$

then

$$(27)\qquad \mathfrak{T} = \begin{pmatrix} m & 0 & 0 & 0 & mc & -mb \\ 0 & m & 0 & -mc & 0 & ma \\ 0 & 0 & m & mb & -ma & 0 \\ 0 & -mc & mb & I_x & -D_z & -D_y \\ mc & 0 & -ma & -D_z & I_y & -D_x \\ -mb & ma & 0 & -D_y & -D_x & I_z \end{pmatrix}$$

is the motor dyadic of inertia.

The most important of the transformations defined by $\mathfrak{T}$ is the transformation of the *motor of velocity* $\mathfrak{V}$ *into the motor of momentum* $\mathfrak{M}$. If a general velocity $\mathfrak{V}$ is given by ω and $\mathbf{v}_p$ where the velocity $\mathbf{v}_p \equiv \mathbf{v}$ of a point p equals

$$(5')\qquad \mathbf{v} = \mathbf{v}_o + (\omega \times \mathbf{r}),$$

we define the *motor of momentum* $\mathfrak{M}$ by its components

$$(28)\qquad \mathbf{M} = \int \mathbf{v}\,dm, \qquad \mathbf{M}_p = \int (\mathbf{r} \times \mathbf{v})\,dm.$$

The integrals being Stieltjes' integrals and dm the element of mass. Then it can be shown that

$$\mathfrak{M} = \mathfrak{T} \cdot \mathfrak{V}.$$

We then obtain the *dynamical equations of the rigid body* in a form *which is independent of the direction as well as of the origin of the coordinate system*:

$$(29)\qquad \frac{d\mathfrak{M}}{dt} = \mathfrak{F}, \quad \text{where} \quad \mathfrak{M} = \mathfrak{T} \cdot \mathfrak{V},$$

or

$$(29')\qquad \frac{d}{dt}(\mathfrak{T} \cdot \mathfrak{V}) = \mathfrak{F},$$

where $\mathfrak{F}$ is the resultant motor of the external forces. The six equations (29′) are the generalization of Newton's "second law of motion" for a particle:

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{f}.$$

It is now easy to write (29′) with respect to any coordinate system which may be fixed in space, moving with the body, or moving independently of the body.

5. A problem in the theory of flight. As an application let us consider the following problem: An *airplane* may be regarded from the kinematical point

of view as a rigid body. The forces on the right side of (29) are: the motor of gravity $\mathfrak{G}$, the propeller thrust $\mathfrak{P}$, and the forces $\mathfrak{A}$ exerted by the air, such as lift and drag, which act on the different parts of the airplane. Let us first consider the left member of (29′). By the use of

$$\frac{d}{dt}(\mathfrak{T}\cdot\mathfrak{V}) = \frac{d'}{dt}(\mathfrak{T}\cdot\mathfrak{V}) + (\mathfrak{V}\times\mathfrak{T}\cdot\mathfrak{V})$$

(29′) becomes since $d'\mathfrak{T}/dt=0$ for a coordinate system fixed in the body,

$$\mathfrak{T}\cdot\frac{d'\mathfrak{V}}{dt} + (\mathfrak{V}\times\mathfrak{T}\cdot\mathfrak{V}) = \mathfrak{G} + \mathfrak{P}. + \mathfrak{A}, \tag{30}$$

As to the right-hand member of (30), the propeller thrust $\mathfrak{P}$ may be assumed as given. Furthermore it is sensible to assume that in (30) the aerodynamical forces $\mathfrak{A}$ depend only or mainly *on the velocity motor* $\mathfrak{V}$ of the plane, *i.e.* the six components of the resultant aerodynamical force $\mathfrak{A}$ are functions of the six components of $\mathfrak{V}$ with respect to a coordinate system which is fixed in the airplane. The gravity motor $\mathfrak{G}$ has obviously a constant direction with respect to a coordinate system at rest, but its line of action is moving together with the airplane, hence

$$\frac{d\mathfrak{G}}{dt} = \frac{d'\mathfrak{G}}{dt} + (\mathfrak{V}\times\mathfrak{G}).$$

Because of the constant direction of $\mathfrak{G}$ we get for its vector-component G

$$0 = \frac{d'\mathrm{G}}{dt} + (\omega\times\mathrm{G}). \tag{31}$$

Hence we have a system of nine differential equations in all for the nine unknowns $\mathfrak{V}$ and G.

An interesting problem now is to consider the *small oscillations* of this system. We assume in the usual way that $\mathfrak{V}_1$ is a possible *stationary* velocity and put $\mathfrak{V}=\mathfrak{V}_1+\mathfrak{V}_2$ where we neglect higher powers of $\mathfrak{V}_2$; let $\mathfrak{A}_1$ be the part of $\mathfrak{A}$ corresponding to $\mathfrak{V}_1$ and put $\mathfrak{A}=\mathfrak{A}_1+\mathfrak{A}_2$ etc. Notice that (30) will be satisfied by $\mathfrak{V}_1$, $\mathfrak{G}_1$, $\mathfrak{P}_1$ and $\mathfrak{A}_1$ and neglect higher powers of $\mathfrak{V}_2$. Then we get the motor equation

$$\mathfrak{T}\cdot\frac{d'\mathfrak{V}}{dt} + \mathfrak{V}_2\times(\mathfrak{T}\cdot\mathfrak{V}_1) + (\mathfrak{V}_1\times\mathfrak{T}\cdot\mathfrak{V}_2) = \mathfrak{G}_2 + \mathfrak{P}_2 + \mathfrak{A}_2 \tag{32}$$

and in the same way

$$\frac{d'\mathrm{G}_2}{dt} + (\omega_1\times\mathrm{G}_2) + (\omega_2\times\mathrm{G}_1) = 0. \tag{33}$$

Here $\mathfrak{A}_2$ is now a *linear* function of $\mathfrak{V}_2$,

$$\mathfrak{A}_2 = \mathfrak{M}\mathfrak{V}_2, \tag{34}$$

i.e. the six components of $\mathfrak{A}_2$ are *linear functions* of the six components of $\mathfrak{V}_2$, and *the whole problem of stability depends on the two motor dyadics* $\mathfrak{T}$ and $\mathfrak{M}$. It would be very interesting to study this problem and to indicate the conditions of stability in terms of *invariants* of these motor dyadics.

6. The three-dimensional particular cases. To finish with a rather elementary remark, let us return to the concept of the dyname $\mathfrak{F}$ as explained in §1. There are three important particular cases which arise when some of the six components in the general dyname vanish. They can be described as follows:

(I) F_1, F_2, F_3 not all zero, $F_4 = F_5 = F_6 = 0$;

(II) F_1, F_2, F_6 not all zero, $F_3 = F_4 = F_5 = 0$;

(III) F_3, F_4, F_5 not all zero, $F_1 = F_2 = F_6 = 0$.

The first case is simply *vector geometry*; the moments vanish because the vector can be translated so as to pass through any point o. The second case corresponds to a *system of bars in the xy-plane* with moments parallel to the z-axis. The third case is illustrated either by a *system of forces parallel to the z-axis* with their moments in the xy-plane, or by a *general rigid motion in the xy-plane* defined by a single bar in the z-direction whose line is the axis of rotation and whose vector is the angular velocity ω, parallel to the z-axis. We thus see that the geometry of forces in the plane is different from the geometry of plane kinematics. The two geometries are however the same in the general three-dimensional case.

May I come back to what I mentioned at the beginning of this lecture: From the point of view of pure mathematics the theory of dynames forms an interesting chapter of geometry. From the point of view of the applications the value of these concepts is that they are helpful in studying the more difficult and general problems of mechanics, just as vector calculus has been for many years, and as tensor calculus is indispensable in the theory of relativity. Moreover, an appropriate calculus often enables us to grasp better the *essential* features of a problem. I should be glad if you got the impression that in this classical field of the geometric foundations of mechanics there are still interesting questions, and that the study of these questions may be worth while for the pure as well as the applied mathematician.

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CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1942-43

Mathematics Club, Brown University

In each of the three semesters two of the meetings of the Club were devoted to a program of two half-hour talks by student speakers, followed by light refreshments. The following twelve papers were presented:

Paper folding, by Rose Mary Canning.

Solutions of the problem of Snellius and Pothenot, by F. J. Hunt, Jr.

Solutions of cubic equations by carpenter squares and by hydrostatics, by Q. B. Leonard.

The Fibonacci series, by J. E. Cook, Jr.

Flatland, by Barbara Cotter.

A problem in life insurance, by W. W. Keffer.

The game of Nim, by D. G. Fernald.

Mathematical fallacies, by B. F. Taylor, Jr.

War problems in mathematics, by Frances Weeden.

Adjustment on the target for artillery fire, by S. L. Ehrlich.

Military and Naval maps, by Ruth Pearson.

Professor Gilman's new method in cryptanalysis, by L. W. Lees, Jr.

Two special meetings were addressed by visiting speakers:

Some properties of systems of circles, by Professor Virgil Snyder.

Third dimension air photographs in this war, by Dr. Richard Goldthwait.

Dr. Goldthwait spoke at the last meeting of the third semester. After this meeting, and after the December meeting at which the annual club picture was taken, some informal dancing was provided in the Crystal Room. Picnics were held in September and May. Chairmen of the Subcommittee on Arrangements (in the semesters indicated) were: J. H. Alger (1, 2 sem.), Q. B. Leonard (3 sem.). Other student members of the Committee on Program and Arrangements were: R. P. Breeding (3), Rose Mary Canning (2, 3), J. E. Cook, Jr. (1), Barbara Cotter (3), Doris Dunlap (1, 2), D. G. Fernald (3), Helen Hooper (1), Arline Major (1, 2), G. A. Levine (1, 2), J. R. Lombardo (2), B. F. Taylor, Jr. (1, 2). Professor R. C. Archibald was Faculty Representative.

Pi Mu Epsilon, Duke University

Four meetings of the fraternity were held this year, at which the following papers were presented:

Paths and Euler's theorem, by Dr. P. T. Maker, Mathematics Department.

Radio and sound, a lecture accompanied by demonstrations by Dr. W. J. Seeley of the School of Engineering.

Sound ranging, by Lt. Col. W. C. Bullock, F. A., Fort Bragg, N. C.

Practical applications of mathematics to astronomy, by Professor K. B. Patterson.

Initiation was held at the third meeting of the year on April 8, and the election of officers was held before Professor Patterson's lecture at the final meeting. Officers for 1943–44 are: President, Bessie Alston Cox; Vice-President, E. H. Smith; Treasurer, William Freeze; Faculty Adviser, Dr. W. W. Elliott.

Mathematics Club, Allegheny College

The club was newly organized in November 1942, and monthly meetings were planned. The topics presented were:

Continued fractions, by Gilbert Michel.

The game of Nim, by James Brown.

Hits and misses, by Professor J. S. Frame.

Diophantine equations, by Joseph Lepore.

Refreshments were served after each meeting. The aftermath of the second meeting, which was held at the home of Professor Frame, was featured by a Nim contest in which the speaker of the evening took on all comers. The evening ended with a showing of some colored movies. Officers for the year were: President, John Caughey; Secretary-Treasurer, Marcella Cooper; Faculty Adviser, Professor J. S. Frame.

Mathematics-Physics Club, College of Saint Teresa

The club meetings of the current year emphasized the role of mathematics in the war program. Reports were given on the following topics:

Mathematics and the Navy

Mathematics and the Army

Geometric design in peace and war

The elements of aeronautics.

A guest speaker, Brother Gerald of Saint Mary's College, Winona, Minnesota, presented this last topic. Other papers presented were:

Oddities in number

Magic squares

Pentagon building

Weather forecasting

Biographies of *Reginald Joseph Mitchell*, *Nicholas Saunderson*, *Newton*, *Steinmetz*, *Heaviside*, *Pupin*, *Karapetoff*, and *Jefferson.*

The Square versus the Circle: a mock debate.

The social meetings consisted of a Valentine party and a mathematical military picnic carried out on the plan of an induction into the *MATHS.* Officers were: President, Mary Brown; Vice-President, Mary Greisch; Secretary, Mary Louise Linehan; Treasurer, Mary Thornton; Faculty Advisers, Sisters Thomas à Kempis and Sister Leontius.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

FOUR POINTS

R. C. YATES, U. S. Military Academy

A conic may be determined by five coplanar points, a rectangular hyperbola by four. No generality is sacrified if the hyperbola be taken as $xy=1$. Thus consider the four points $(\lambda_i, 1/\lambda_i)$ on $xy=1$ and let $\lambda_1\lambda_2\lambda_3\lambda_4=\sigma$. The orthocenter of the triangle $(i=1, 2, 3)$ is the intersection of the altitudes

$$\lambda_2\lambda_3 x - y = \lambda_1\lambda_2\lambda_3 - 1/\lambda_1$$

$$\lambda_3\lambda_1 x - y = \lambda_1\lambda_2\lambda_3 - 1/\lambda_2$$

that is, the point $(-1/\lambda_1\lambda_2\lambda_3, -\lambda_1\lambda_2\lambda_3)$. Accordingly, three points on a rectangular hyperbola have their orthocenter also on the curve. This is a well known property. The four chosen points are orthocentric if

$$\sigma = -1.$$

Any circle $x^2+y^2-2ax-2by+c=0$ cuts the hyperbola $xy=1$ in points determined by $x^4-2ax^3+cx^2-2bx+1=0$. Thus if the four points are cyclic,

$$\sigma = +1.$$

This is also sufficient since the determinant $|\lambda_i^2+1/\lambda_i^2, \lambda_i, 1/\lambda_i, 1|$, $(i=1, 2, 3, 4)$, vanishes if $\sigma=+1$.

We return now to the original set of four points $(\lambda_i, 1/\lambda_i)$. Taken in sets of three, their orthocenters are $(\mu_i, 1/\mu_i)$, where

$$\mu_i = -\lambda_i/\sigma.$$

We ask that this set of orthocenters $(\mu_i, 1/\mu_i)$ fall coincident in the order (1234) on (1234) with the selected set $(\lambda_i, 1/\lambda_i)$. Then $\sigma=-1$, which is the condition that $(\lambda_i, 1/\lambda_i)$ be orthocentric.

If, instead, the μ's fall on the λ's in the order (1234) on (2143), we have

$$\mu_1=\lambda_2=-\lambda_1/\sigma;\quad \mu_2=\lambda_1=-\lambda_2/\sigma;\quad \mu_3=\lambda_4=-\lambda_3/\sigma;\quad \mu_4=\mu_3=-\lambda_4/\sigma.$$

This is possible in either of two ways: first, if $\sigma=-1$ and $\lambda_1=\lambda_2$, $\lambda_3=\lambda_4$; second, if $\sigma=+1$ and $\lambda_1=-\lambda_2$, $\lambda_3=-\lambda_4$. This latter arrangement is of interest. Here the points $(\lambda_i, 1/\lambda_i)$ lie on a circle with center at the origin. In the cyclic order (1423) they form a rectangle with area

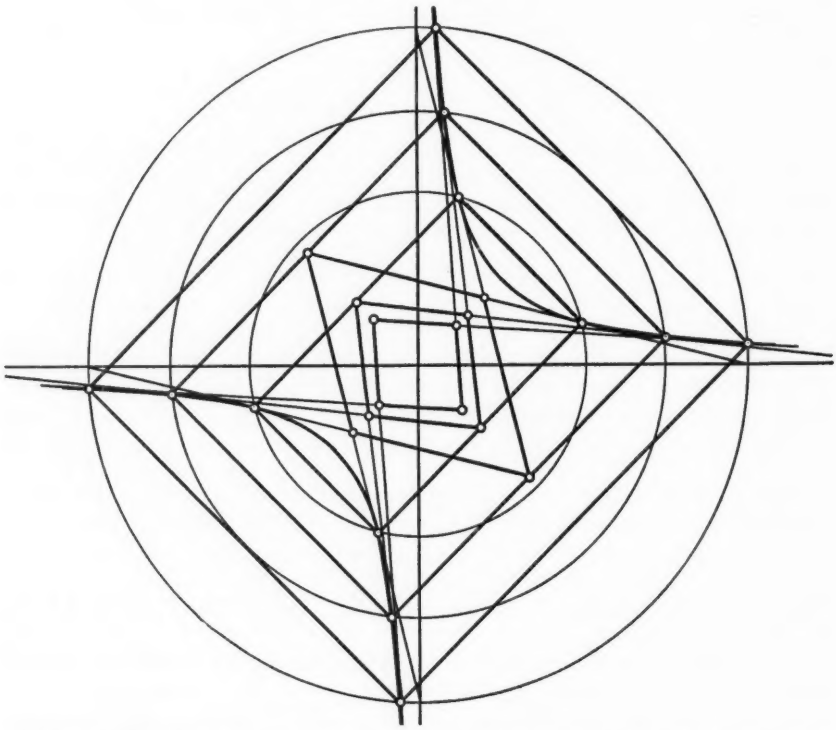
$$A = 2\cdot|\lambda_1^2-\lambda_3^2|.$$

The tangents to the hyperbola at these points form a rhombus with area

$$B = 16 \cdot \left| \lambda_1\lambda_3/(\lambda_1^2 - \lambda_3^2) \right|.$$

Thus, since $\sigma = \lambda_1^2\lambda_3^2 = 1$,

$$A \cdot B = 32.$$



The geometrical aspects are now apparent. A fixed rectangular hyperbola intersects a concentric variable circle in four points. The product of the areas of the rectangle on these four points and the rhombus formed by their tangents to the hyperbola is constant.

The following problems to which the preceding notation is particularly well adapted were brought to my attention by Colonel Harris Jones.

If a circle and rectangular hyperbola (not necessarily concentric) meet in four real points: (1) the centroid of the quadrilateral on the four points is the midpoint of the line segment joining the center of the circle and the center of the hyperbola; (2) the sum of the squares of the distances of the four points from the center of the circle equals the sum of the squares of their distances from the center of the hyperbola.

AN ELEMENTARY LIMIT

M. S. KNEBELMAN, State College of Washington

The proof of the fundamental theorem

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1,$$

as ordinarily given in elementary books, usually depends on two unproved theorems. The following proof is at least simpler, if not more rigorous.

If P is the perimeter of a regular n-gon inscribed in a circle of radius r, then $P = 2nr \sin \pi/n$ and we know from plane geometry that $\lim_{n \to \infty} P = 2\pi r$. Hence $\lim_{n \to \infty} (n/\pi) \sin \pi/n = 1$ and if we let $\pi/n = \theta$ then $\theta \to 0$ as $n \to \infty$ and conversely. Hence $\lim_{\theta \to 0} \sin \theta/\theta = 1$.

QUADRATIC AND CUBIC EQUATIONS*

O. J. RAMLER, Catholic University of America

1. Introduction. When an algebraic equation having real coefficients is subjected to a bilinear transformation which transforms the axis of reals in the complex plane into the unit circle, the new equation will generally have complex coefficients and the roots of the original equation will either lie on the unit circle or pair off as points inverse to the unit circle. It would therefore seem reasonable to expect that the nature of the roots with respect to the unit circle of an algebraic equation with complex coefficients should be readily obtained by applying the bilinear transformation mentioned. However, even in the simple case of the quadratic equation this method of approach to the problem becomes long and involved. The following treatment is simpler and so far as the writer is aware does not appear in the literature.

2. The quadratic equation. Consider the quadratic equation

$$z^2 + pz + q = 0. \tag{1}$$

Let p and q have moduli r_1 and r_2 and amplitudes θ_1 and θ_2 respectively. Then we may write (1) in the form

$$z^2 + r_1 t_1 z + r_2 t_2 = 0,$$

where $t_1 = e^{i\theta_1}$ and $t_2 = e^{i\theta_2}$. The equations whose roots are the conjugates and the reciprocals of the roots of (1) are, respectively,

$$t_1 t_2 z^2 + r_1 t_2 z + r_2 t_1 = 0$$

and

$$r_2 t_2 z^2 + r_1 t_1 z + 1 = 0.$$

* Presented to the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America at Ashland, Virginia, May 2, 1942.

The equivalence of these two equations is the necessary condition that the roots of (1) shall lie on the unit circle. This condition leads to $r_2=1$ and $t_2=t_1^2$. Then equation (1) takes the form

$$z^2 + rtz + t^2 = 0,$$

whose roots will be denoted by z_1 and z_2. When $r>2$, $z_1\bar{z}_2=1$, and the roots are inverse points with respect to the unit circle. When $r<2$, $z_1\bar{z}_1=1$, and $z_2\bar{z}_2=1$, showing that z_1 and z_2 lie on the unit circle. Summarizing, we have

THEOREM 1. *The roots of $z^2+pz+q=0$ lie on the unit circle when the clinant $p/\bar{p}$ of p is q and $|p|<2$. The roots are inverse to the unit circle when the clinant of p is q and $|p|>2$. The roots are coincident points on the unit circle when $|p|=2$ and the clinant of p is q.*

3. The cubic equation. Let the cubic equation be

$$z^3 + at_1z^2 + bt_2z + ct_3 = 0, \tag{2}$$

where a, b, and c are positive and real and $|t_1| = |t_2| = |t_3| = 1$. Comparing the equations whose roots are the conjugates and the reciprocals of the roots of equation (2), we find the necessary conditions that the roots of (2) shall lie on the unit circle to be $a=b$, $c=1$, and $t_1t_2=t_3$. Hence (2) may be written

$$z^3 + at_1z^2 + at_2z + t_1t_2 = 0. \tag{3}$$

Let the roots of (3) be x_1, x_2, x_3. If $x_1\bar{x}_1=x_2\bar{x}_2=x_3\bar{x}_3=1$, the roots are all obviously on the unit circle and $a\leqq 3$. If $x_1\bar{x}_2=x_2\bar{x}_3=x_3\bar{x}_1=1$, the roots are inverse to the unit circle in pairs. Hence $x_1=x_2=x_3$, $a=3$, and $t_2=t_1^2$. If $x_1\bar{x}_1=x_2\bar{x}_3=\bar{x}_2x_3=1$, x_1 will lie on the unit circle and x_2 and x_3 will be inverse with respect to it. Thus we find that at least one of the roots of (3) must lie on the unit circle and the other two lie on it or are inverse to it.

The roots of the Hessian of the cubic are the Isodynamic or Hessian points of the triangle determined by the cubic [1]. The Hessian points lie on the Apollonian circles of the triangle [2]. When the triangle determined by the cubic (3) is inscribed in the unit circle the Hessian points are inverse to the unit circle. When the cubic represents one point on the unit circle and two points inverse with respect to it the circumcircle of the triangle of roots will be orthogonal to the unit circle. Moreover in this case, the unit circle is an Apollonian circle of the triangle. Hence when the roots of the cubic represent one point on the unit circle and a pair of points inverse to it, the roots of the Hessian must lie on the unit circle.

The Hessian of (3) is

$$H = a(3t_2 - at_1^2)z^2 + (9 - a^2)t_1t_2z + at_2(3t_1 - at_2).$$

Identifying $H=0$ with $z^2+pz+q=0$ we readily find that $p/\bar{p}=q$ which insures

its roots to be either on the unit circle or inverse points with respect to it. From $H=0$ we have

$$|p| = \frac{|9 - a^2|}{a|3t_2 - at_1^2|}.$$

Hence

$$|3 - a|/a \leqq |p| \leqq (3 + a)/a,$$

and the restrictions on a and p may be analyzed.

The results for the cubic equation (3) may then be summarized in

THEOREM 2. *One root lies on the unit circle and two roots are inverse to it if* $a>1$ *and* $|p|<2$ *unless* $a=3$, $p=0$, *and* $t_2=t_1^2$ *when all three roots are coincident on the unit circle and the Hessian vanishes identically. All roots lie on the unit circle forming*

an equilateral triangle if $a=0$, $p=\infty$;
an acute-angled triangle if $0<a<1$, $|p|>2$;
an isosceles right triangle if $a=1$, $|p|=4$;
a scalene right triangle if $a=1$, $2<|p|<4$;
an obtuse triangle if $1<a<3$, $|p|>2$.

If $a=1$ *and* $|p|=2$, *two roots are coincident at one end of a diameter and the third root is at the other end of the diameter. If* $1<a<3$ *and* $|p|=2$, *the cubic and its Hessian have the same root repeated; the third root is distinct but lies on the unit circle.*

REFERENCES

1. F. V. and F. Morley, Inversive Geometry, p. 81 and p. 209.
2. R. A. Johnson, Modern Geometry, §492.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

A Source Book of Mathematical Applications. By E. G. Olds, L. E. Boyer, Ruth O. Lane, Nathan Lazar and F. L. Wren. (Seventeenth Yearbook of the National Council of Teachers of Mathematics.) New York, Bureau of Publications, Teachers College, Columbia University, 1942. 16+291 pages. $2.00.

This book is intended to be a reference book for those teaching junior and senior high school mathematics. Its four sections contain a wide variety of illus-

trations of the application of Arithmetic, Algebra, Geometry and Trigonometry to problems in daily life, in the professions and in various trades. These illustrations range from discussions or mere suggestions to the actual statement and solution of problems. The book is well arranged and indexed so that quick reference can be made to illustrations requiring a particular mathematical skill or to those pertaining to a given occupation. Most professions and trades are well represented, but it is surprising to find that the field of Chemistry is hardly mentioned.

Many unusual sources are drawn upon; yet the principal sources listed, especially in the Geometry section, are other mathematics text-books. It seems superfluous to acknowledge the permission of a well-known motor company for the use of Euler's formula for the lowest critical load of a column, and for the ordinary formula for the deflection of a cantilever.

One would expect from a mathematics book a far higher standard of precision than is found here. How can mathematical accuracy be represented by a discussion of astronomical data which begins with numbers given to two or four significant figures and derives from them numbers containing nine and 21 significant figures? The last number mentioned undoubtedly makes a very impressive addition to the book. We meet statements such as "$60\times10\times6=\$0.036$," "$240\times.0013=.312$ ft.," or "a gun has an initial velocity of 1500 feet per second." We are told that "the earth, all planets, and all satellites except a very few have their revolution about the sun and their rotation on their axis in a counter-clockwise direction."

Often units to be used are not specified, and numerous technical details must be supplied before such examples are used. Perhaps the authors hoped to make up for this by carefully specifying units in nearly all of their problems on variation—where they do no good at all because not a single set of data is given from which a constant of variation can be found. Many quantities such as work and power, force and pressure, velocity and acceleration are sometimes confused. Because of this careless inattention to units, a large portion of the Algebra section is of questionable value as reference material. This book should not be used by any teacher who lacks a working knowledge of physics.

R. E. GASKELL

An Introduction to Analytic Geometry. Volume I. By A. Robson. Cambridge, University Press, 1940. 14+409 pages. $2.50.

This book starts at the beginning, but really presupposes a knowledge of graphs and of the elementary processes of the calculus, as the new ideas are introduced rapidly. Vectors and scalars, polar coordinates, determinants, homogeneous coordinates are treated at once; graphs and loci, points of intersection, change of axes are all discussed before the equation of a straight line is presented. The concepts of duality and of envelopes appear very early, together with those of composite loci and composite envelopes. The chapter on the circle

includes a discussion of pencils of circles, power of a point, radical axis, and limiting points.

True to the English tradition, generous lists of exercises for the reader to solve are found throughout the book. The explanations are concise and brief, but it is supposed that the added exercises will clarify the meaning of the text. A chapter on parametric representation follows immediately that on the circle. The discussion includes both point and line coordinates, with conditions for collinearity and concurrence. A short chapter on algebraic curves and one on abstract geometry introduce geometry of the complex domain. The presentation is interesting and is carefully discussed. It seems to the reviewer to be too brief to be of greatest service.

Now follow a number of concepts from pure geometry, though clothed for the most part in algebraic form. These are conical projection, cross-ratio, harmonic section, Desargues's theorem on perspective triangles, pole and polar, the general conic in point and in line coordinates, and the self-polar triangle.

Finally, after all these there is a series of chapters on the parabola, ellipse, and hyperbola, in which the ordinary topics usually associated with these names are treated in great detail. The book closes with four long lists of further exercises arranged in order of increasing difficulty. Answers to all the exercises are provided. There is a full index. The press-work is strikingly clear and legible. The figures are excellent. They are unnumbered.

A reader who has mastered this text and has solved a considerable number of the exercises has an adequate knowledge of both analytic and projective plane geometry.

VIRGIL SNYDER

NEW BOOKS RECEIVED

First Year College Mathematics. By W. L. Hart, W. A. Wilson, and J. I. Tracy. Fourth Edition. Boston, D. C. Heath and Co., 1943. 7+279+2+124 pages. $4.00.

Differential and Integral Calculus. Fourth Edition. By C. E. Love. New York, The Macmillan Co., 1943. 15+483 pages. $3.25.

Lecciones de Geometria Analitica. By Guido Castelnuovo. Translated by Andrea Levialdi and Manuel Sadosky, from the 7th Italian edition. LaPlata, Editorial Mundo Cientifico, 1943. 7+657 pages.

Plane and Spherical Trigonometry. By A. L. Nelson and K. W. Folley. With Mathematical Tables. Revised edition. New York, Harper and Brothers, 1943. 14+247+135 pages. $2.40.

Student's Handbook of Elementary Physics. By R. B. Lindsay. New York, The Dryden Press, 1943. 15+382 pages. $2.25.

Business Mathematics. By C. C. Richtmeyer and J. D. Foust. Second edition. New York and London, McGraw-Hill Co., 1943. 15+401 pages. $2.75.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 586. *Proposed by Frank Hawthorne, Allegheny College*

What is the path of an airplane "homing" from A to B by directional radio from B only, if acted on by a constant wind?

E 587. *Proposed by V. Thébault, San Sebastián, Spain*

Let AA', BB', CC', DD' be the altitudes of an orthocentric tetrahedron $ABCD$, with orthocenter H. Show that

$$\frac{BC\cdot DA}{B'C'\cdot D'A'}=\frac{CA\cdot DB}{C'A'\cdot D'B'}=\frac{AB\cdot DC}{A'B'\cdot D'C'}=\frac{HA\cdot HB}{HC'\cdot HD'}=\frac{HC\cdot HD}{HA'\cdot HB'}=\cdots.$$

E 588. *Proposed by J. Rosenbaum, Bloomfield, Connecticut*

Solve the equation

$$x^4-2cx^3+(2c^2-r^2)x^2+2cr^2x-c^2r^2=0.$$

When will two of the roots be rational? Can all four roots be rational?

E 589. *Proposed by Lloyd Dulmage, University of Manitoba*

Let ${}_2H_n$ denote the number of ways of putting n letters and n checks into n envelopes (respectively) so that every envelope contains either a wrong letter or a wrong check (or both), and let ${}_2J_n$ denote the number of ways when each envelope contains both a wrong letter and a wrong check. Show that

$${}_2H_n=\Delta^n(0!)^2,\qquad {}_2J_n=(\Delta^n 0!)^2,$$

and generalize to ${}_sH_n$, ${}_sJ_n$.

E 590. *Proposed by H. W. Becker, Mare Island Training Division*

Let ${}_sK_n$ denote the number of Latin rectangles of n columns and $s+1$ rows, the first row consisting of the n letters in their natural order. Also let L_n denote the total number of $n\times n$ Latin squares (so that $L_2=2$, $L_3=12$, $L_4=576$,

$L_5 = 161280$, $L_6 = 812851200$, and $L_7 \geqq 61428210278400$). Show that, for a given s,

$$_sK_n \sim (n!)^s e^{-s(s+1)/2} \quad \text{as} \quad n \to \infty,$$

and that

$$L_n \sim (n!)^n e^{-n(n+1)/2}.$$

SOLUTIONS

The Meigs Hall Problem

E 531 [1942, 475; 1943, 202]. *Proposed by P. R. Hill, University of Georgia*

Suppose six students be standing an examination in a row of seats with an aisle at each end. If they finish in random order, what is the probability that a student will have to pass over one or more other students in order to reach an aisle? (The proposer intended "a student" to mean a particular student whose seat is not specified. Previous solvers took it to mean "some student.")

III. *Solution by W. H. Markham, John McNeese Junior College of the Louisiana State University.* The probability of any student finishing first is 1/6; the probability that the student finishing first will have to pass one or more students is 2/3; therefore the probability of any student finishing first and having to pass one or more students is $(1/6)(2/3) = 1/9$. Similarly, the probability of any student finishing second and having to pass one or more students is $(1/6)(3/5) = 1/10$; and the probability of any student finishing third or fourth and having to pass is $(1/6)(1/2) = 1/12$ or $(1/6)(1/3) = 1/18$, respectively. There will be no occasion for a student finishing fifth or sixth to pass any others. Hence the probability that any student will have to pass one or more students in order to reach an aisle is

$$\tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{12} + \tfrac{1}{18} + 0 + 0 = \tfrac{7}{20}.$$

Also solved by E. P. Starke and the proposer.

Perfect Squares with Even Digits

E 548 [1942, 683]. *Proposed by R. V. Heath, Wall St., New York City*

Find a perfect square of seven digits with all digits even and positive. Show that the digits of a perfect square (>9) cannot be all *odd*.

Solution by Alan Wayne, Flushing, New York. The last two digits of N determine the last two digits of N^2. If e, o, a denote even, odd, any integers, respectively, then it is easily seen that

$$(eo)^2 = aeo, \qquad (oo)^2 = aeo, \qquad (ee)^2 = aae, \qquad (oe)^2 = aae,$$

whence the ultimate and/or penultimate digit must be even.

A scrutiny of Barlow's Tables gives

$$1692^2 = 2862864, \qquad 2878^2 = 8282884, \qquad 2978^2 = 8868484.$$

Examples of eight-digit squares of the same type are:

$$4738^2 = 22448644, \quad 5162^2 = 26646244, \quad 6668^2 = 44462224,$$
$$9068^2 = 82228624, \quad 9092^2 = 82664464.$$

The only five-digit squares with all even digits (allowing zero) are the following five:

$$162^2 = 26244, \quad 168^2 = 28224, \quad 202^2 = 40804, \quad 262^2 = 68644, \quad 298^2 = 88804.$$

The only perfect squares which have all digits even and different are $8^2=64$ and $78^2=6084$.

Also solved by R. K. Allen, W. E. Buker, M. L. Constable and William McGavock.

Note. One of the above eight-digit squares generalizes to any even number of digits:

$$(6\cdots68)^2 = 4\cdots462\cdots24.$$

Sums of Even Powers of Alternate Numbers

E 550 [1942, 683]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Prove that, for every positive even value of m, the sum of the mth powers of alternate numbers from 1 or 2 up to n is a polynomial function of n (of the same form whether n be odd or even).

Solution by Irving Kaplansky, Harvard University. We assume the well known result that $1^m+2^m+\cdots+n^m$ is a (necessarily unique) polynomial $f(n)$. The relation

$$f(n) - f(n-1) = n^m \tag{1}$$

is an identity since it holds for infinitely many values of n. So also is the consequent relation

$$f(n) - f(n-r) = n^m + (n-1)^m + \cdots + (n-r+1)^m.$$

Putting $n=0$, we deduce (for $m>0$)

$$-f(-r) = (-1)^m f(r-1),$$

which again is an identity (since it holds for infinitely many values of r). It follows that, when m is even and positive, $f(-\frac{1}{2})=0$.

For n even,

$$2^m + 4^m + \cdots + n^m = 2^m f(n/2);$$

and for n odd,

$$1^m + 3^m + \cdots + n^m = f(n) - 2^m f\left(\frac{n-1}{2}\right).$$

We wish to establish the vanishing of the difference

$$g(n) = f(n) - 2^m f\left(\frac{n-1}{2}\right) - 2^m f\left(\frac{n}{2}\right).$$

By (1), we have

$$g(n) - g(n-1) = n^m - 2^m(n/2)^m = 0.$$

Therefore, for m even and positive,

$$g(n) = g(n-1) = \cdots = g(0) = -2^m f(-\tfrac{1}{2}) = 0.$$

Editorial Note. The Euler-Maclaurin summation formula

$$\Delta^{-1}u_x = \int u_x dx + \sum_{r=1}^{\infty} \frac{B_r}{r!}\left(\frac{d}{dx}\right)^{r-1} u_x$$

[cf. 1943, 60] gives

$$\Delta^{-1}(2x)^m = \frac{2^m}{m+1}\left\{x^{m+1} + \sum_{r=1}^{m+1}\binom{m+1}{r} B_r x^{m-r+1}\right\} + C$$

$$= \frac{2^m}{m+1}(B+x)^{m+1} + C$$

(using B^r for B_r). Hence the sum of mth powers of alternate numbers from 1 or 2 to n is

$$n^m + \frac{2^m}{m+1}(B+x)^{m+1}\Big|_{x=1/2 \text{ or } 1}^{x=n/2},$$

where the lower limit is $\frac{1}{2}$ or 1 according as n is odd or even. If this expression is the same in either case, we have

$$(B+\tfrac{1}{2})^{m+1} = (B+1)^{m+1} = B_{m+1}.$$

But B_{m+1} vanishes for all positive even values of m. Kaplansky's argument thus provides a new proof for the known relation

$$(B+\tfrac{1}{2})^k = 0 \qquad (k \text{ odd}),$$

which De la Vallée Poussin (*Cours d'Analyse Infinitésimale*, vol. II, Louvain, 1925, p. 336) deduces from the symbolic formula

$$e^{Bx} = x/(e^x - 1)$$

as follows. We have

$$e^{(B+1/2)x} + e^{Bx} = e^{Bx}(e^{x/2}+1) = x/(e^{x/2}-1) = 2e^{Bx/2},$$

whence, equating coefficients of x^k,

$$(B+\tfrac{1}{2})^k + B_k = 2(B/2)^k = B_k/2^{k-1}.$$

Incidentally, we have found (for any positive even value of m) the expression

$$f(n) = n^m + \frac{(n + 2B)^{m+1}}{2(m + 1)}$$

(in which, after expanding by the binomial theorem, we replace the symbols $B, B^2, B^3, B^4, \cdots$ by $-\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, \cdots$).

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4093. *Proposed by B. M. Stewart, Michigan State College*

A store conducted the following game in which the player paid 10¢, and if he won received \$1 in merchandise. Out of five throws, each time throwing ten dice, a win was declared if the player had *fourteen* or more appearances of the numbered face named by him before playing; the player had the additional advantage that on the *first* throw of ten dice he could count the face occurring the greatest number of times as if it were the special face he had selected. The problem is to compare the theoretical probability with the odds offered by the store.

4094. *Proposed by N. A. Court, University of Oklahoma*

A. If a line moves so that the segment intercepted on it by two fixed skew lines subtends a right angle at a fixed point in space, the line generates a ruled quadric (This proposition is due to E. Bobillier, 1797–1832. It was proved analytically in *Nouvelles Annales de Mathématiques*, 1862, pp. 318, 320 and 379, 380).

B. The segment intercepted by two elements of the same system of a ruled quadric on a variable element of the complementary system subtends a right angle at each of two fixed points in space.

4095. *Proposed by H. D. Ruderman, Cooper Jr. H. S., Manhattan, N. Y.*

If x_1 and x_2 are any two real roots of the equation

$$a \sin x + \sin (bx + c) = 0,$$

with a, b, c real and positive, such that no real root lies between x_1 and x_2, determine the maximum and minimum values of $|x_1 - x_2|$ in terms of a, b, c.

4096. *Proposed by V. Thébault, San Sebastián, Spain*

An octahedron $ABCDEF$ with triangular faces is such that the vertices A, B, C, D are coplanar. Show that: (1) The intersections of the planes of the opposite faces (BEC, DFA), (BEA, DFC) lie in the same plane (P); the intersections of the planes of the opposite faces (AED, CFB), (DEC, BFA) lie in the same plane (Q); and the intersections of the planes (AEC, DFB), (BED, CAF) lie in the same plane (R). (2) The planes (P), (Q), (R) intersect in the same straight line which meets the diagonal EF.

SOLUTIONS

Artzt Parabolas

4019 [1942, 64]. *Proposed by Robin Robinson, Dartmouth College*

Given a triangle ABC. Prove that the bisectors of the interior and exterior angles at C, the side AB and its perpendicular bisector, and the perpendiculars to AC at A and to BC at B, are all tangent to a parabola. Locate its focus.

II. *Solution by L. M. Kelly, Coast Guard Academy, New London, Conn.* Let angle A of the triangle ABC be less than angle B; let the perpendicular bisector of AB meet the latter in M and the circumcircle (O) in D and H so that CD and CH are the interior and exterior bisectors of angle C; let CO meet (O) again in E; and let G be the projection of D on CA. We show first that GM is parallel to HC. Since G, A, D, M, are concyclic, $\angle MGC = \angle MDA = \angle HCA$, and this proves the statement. Consider now the hexagon $AMDC\,\infty_1\infty_2$, where the fifth and sixth points are at infinity on HC and GM, and on AE and GD. The diagonals are CA, GM, GD, and hence the sides of the hexagon are tangent to a parabola. A similar proof shows that BE is tangent to the same parabola. The directrix is CM, since tangents from C and M are perpendicular. Reflect CM in CD and then in MB; the intersection of these two reflections is the focus.

Solved also by E. P. Starke using the two bisectors of angle C as axes of coordinates. Solution I by the proposer is given [1943, p. 128].

Editorial Note. There exists a parabola tangent to the sides of the quadrilateral formed by the straight lines of HC, CD, HD, AB, and also a parabola tangent to the sides of the one formed by AE, BE, HD, AB. It is known that the focus in each case lies on (O); and, assuming as above that angle A of ABC is less than angle B, the focus in each case lies on the arc $\overset{\frown}{DB}$, where the order of points is D, B, C, H. In the second case, if C' is the orthocenter of AEB, the figure $AC'BC$ is a parallelogram and C' is on the extension of CM; and it follows that in the two cases the directrix is along CM. Let CM meet (O) again in M', and draw $M'F$ parallel to AB meeting the arc $\overset{\frown}{DB}$ in F. Then in each case the reflection of CM in AB (or HD) is along MF; thus F is the focus in each case. Hence we have a construction for the focus F and the directrix CM of the unique parabola tangent to the straight lines of the problem CD, HC, AB, HM, AE, BE.

The following information was supplied recently by J. R. Musselman. The parabola of the problem is related to one of the Artzt parabolas for a triangle ABC which is tangent to the sides of the complete quadrilateral formed by the internal and external bisectors of the angle C and the perpendicular bisectors of sides CA and CB. Its focus is the vertex of the second Brocard triangle for ABC corresponding to C and hence the focus is the midpoint of the chord of circumcircle (O) cut off from the symmedian for C; its directrix is the median for C.

We shall state a proof supposing that angle A is less than angle B. The quadrilateral contains two right triangles RSC, PQC, where the two straight lines QRC, PCS are the internal and external bisectors of angle C; and their circumcircles (RSC), (PQC) intersect again in the focus F. We easily show that these two circles are respectively tangent at C to CA, CB and pass through B, A. Hence F has all of the above properties, see Johnson's *Modern Geometry*, p. 279.

This Artzt parabola is identical with the parabola of the problem, as we see by extending the sides CA, CB of the problem to double their lengths forming the triangle $\overline{AB}C$ for the Artzt parabola. Thus the solution of the problem adds two tangents to the Artzt parabola, the straight line joining the midpoints of the two sides through C and the perpendicular bisector of the segment between these two midpoints.

There is another Artzt parabola which is defined as tangent to CA and CB at A and B. It is easily seen that its axis is parallel to the median for C, and that the straight line joining the midpoints of CA and CB is tangent to the parabola at the midpoint of the segment between these two sides. The focus F lies on the circumcircle of the triangle formed by C and these two midpoints. Since triangles FCB and FAC are similar, the focus F must lie on the symmedian for C; hence it is the midpoint of the symmedian chord of the circumcircle (ABC).

Isogonol Conjugate Points

4038 [1942, 341]. *Proposed by V. Thébault, San Sebastián, Spain*

The point M is chosen arbitrarily on a bisector of angle A of the triangle ABC, and let M' be its isogonal conjugate with respect to ABC. Show that the two circles each through M and M' and tangent to the side BC are tangent also to the circumcircle of ABC.

I. *Solution by Howard W. Eves, Syracuse University.* Let the bisector of angle A meet BC in N. Inverting with respect to B as center of inversion, we obtain the triangle $A'B'C'$; its circumcircle $(A'B'C')$, the inverse of AC; and the circle $(B'N'M''M'A')$, where M and M' invert into M' and M'', and the latter circle bisects the angle formed by $A'B'$ and $(A'B'C')$. We are to prove that a circle through M'' and M' and touching the straight line $B'C'$ touches also the straight line $A'C'$, which is the inverse of the circumcircle (ABC). But this is evident since it is easy to show that the center of $(B'N'M''M'A')$ lies on the midpoint of one of the arcs $A'B'$, and hence on one of the bisectors of angle C'. Since the

arcs $N'M''$ and $M'A'$ are equal, it follows that M'' and M' are symmetric in the bisector of angle C', and the theorem follows.

II. *Solution by L. M. Kelly, U. S. Coast Guard Academy, New London, Conn.* The following equivalent theorem will be demonstrated:

If a circle tangent to the circumcircle of triangle ABC and to its side BC, cuts the bisector of angle A in real points, these points are isogonal conjugates.

The circle (Q) is tangent to the circumcircle (O) of ABC at D, tangent to BC at E, and meets the bisector of angle A in the points M, M'. Let the straight line DE meet (O) again in F; we shall show that F is the midpoint of arc BC of (O) and thus lies on AMM', the bisector. If the two tangents to (Q) at D and E meet in G, the angles GDE and DEG are equal, and arc DBF=arc DB +arc FC. Hence arc BF=arc FC, and the above statement is proved. We show next that triangles BFM' and MFB are similar. They have the angle at F in common and $FM' \cdot FM = FE \cdot FD$; also $FE \cdot FD = FB^2$, since triangles BFE and DFB are similar. Hence $FB^2 = FM' \cdot FM$, and the statement is proved. It follows that angles FBM' and BMF are equal; then angle FBC+angle CBM'=angle BAF+angle MBA. Thus angles CBM' and MBA are equal, and M and M' are isogonal conjugate points.

Editorial Note. The proposer stated that the theorem of this problem is a variation of Bricard's Theorem II in the solution of 3852 [1941, 70]; that its proof is simple by an inversion; and that it is also a generalization of 3887 [1938, 482].

Omitting the parts not required here Bricard's proof of his Theorem II gives a simple and elegant solution of the present problem which we shall state. Let (O) be the circumcircle of ABC; ω, ω_a, ω_b, ω_c the incenter and excenters for angles A, B, C; the interior bisector of angle A contains ω and ω_a and meets BC and (O) in N and I; and the exterior bisector of A contains ω_b and ω_c and meets BC and (O) in N' and I'. It is known and it is easily proved that I is the center of a circle (I) through ω, ω_a, B, C; and similarly for I'. Let F, F' be isogonal conjugate points with respect to ABC on the interior bisector AI, then the straight line $C\omega$ bisects the angle $F'CF$; and it follows that F, ω, F', ω_a is a harmonic set. Any circle (Γ) through F and F' is orthogonal to (I) and is invariant in an inversion with respect to (I); and, if (Γ) is also tangent to BC at T, the inversion carries (Γ), BC, (O), T into (Γ), (O), BC, T', where T' must be the point of tangency of (Γ) with (O). Conversely, if (Γ) passes through F and F' and is tangent to (O) at T', it is tangent also to BC at T, where T', T, I are collinear. If F, F' are isogonal conjugates on AI', we have a similar theorem and proof.

Suppose now that (Γ) is any circle tangent to BC at T and to (O) at T'; then by the first part of Kelly's proof the straight line TT' meets (O) again at I, or at I', and it follows from the above inversion that (Γ) is invariant with respect to (I) in the first case or to (I') in the second. If then in the first case (Γ) meets AI in real points F, F', they are isogonal conjugates; and similarly for the sec-

ond case. We consider now the cases in which TT' passes through I giving possible isogonal conjugates F, F' on AI. The point T lies within the segment BC and T' lies on the arc CAB of (O), and F, F' lie within segment AN; the point T lies on BC outside the segment BC and T' lies on arc BIC giving possible points F, F' on the extension of AI. For the cases where TT' passes through I', we have T outside BC and T' on arc CAB giving possible points F, F' on the extension of AI', or on the extension of $I'A$. There are eight special cases of (Γ) where F, F' coincide at ω, ω_a, ω_b, ω_c which are easily constructed. These special circles (Γ) determine the regions for T on BC, and regions for T' on (O) for which there are no real corresponding points F, F'. For the case of circles (Γ) tangent to BC at T within that segment and to (O) at T' within the arc BIC, the straight line TT' must go through I'; and there are such circles which meet AI in real points, but which cannot be isogonal conjugates.

Tetrahedrons, Isogonal Conjugates

4039 [1942, 341]. *Proposed by N. A. Court, University of Oklahoma*

The circumcenter of a tetrahedron (T) and any point M are isogonal conjugates with respect to the tetrahedron formed by the centers of the four spheres passing through M and the circumcircles of the faces of (T).

Solution by Howard Eves, Syracuse University. The theorem is an obvious consequence of the sufficiency part of the following criterion for two points to be isogonal conjugates with respect to a given tetrahedron.

Criterion: A necessary and sufficient condition for two points to be isogonal conjugates with respect to a given tetrahedron is that one of the points be the circumcenter of the tetrahedron whose vertices are the reflections of the second point in the faces of the given tetrahedron.

The necessity is established in art. 750 of Altshiller-Court's *Modern Pure Solid Geometry*. We now prove the sufficiency. Let the two points be Q and P, and let the reflections of P in any two selected faces ABC and ABD of the given tetrahedron $ABCD$ be D' and C' respectively. Then we have

$$(1) \qquad D' - AB - C = C - AB - P, \quad P - AB - D = D - AB - C'.$$

Also, since Q and AB are each equidistant from C' and D',

$$(2) \qquad D' - AB - Q = Q - AB - C'.$$

But

$$D'-AB-C+C-AB-P+P-AB-D+D-AB-C'=D'-AB-Q+Q-AB-C',$$

whence, by (1) and (2),

$$C - AB - P = Q - AB - C' - D - AB - C' = Q - AB - D.$$

It may similarly be shown that P and Q form equal dihedral angles with every other pair of faces of $ABCD$. This proves the sufficiency of the criterion.

The corresponding theorem for the plane may be established in an analogous manner.

Solved also by L. M. Kelly and the proposer.

Editorial Note. The remaining two solutions are similar and consist of the argument for the part of the above solution which leads to its criterion, but using in place of it the theorem of art. 750 of the cited text. The converse of this latter theorem is easily proved by reversing the proof given in the text. The isogonal conjugate relation between two points ceases to be of one to one character if one of the points lies in the plane of a face of the given tetrahedron, and in this case one or both of the two tetrahedrons obtained by reflection degenerates, two or three of the vertices coinciding. However, the theorem of the problem remains true in this case of degeneracy.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Professor Alfonso Nápoles of the National University of Mexico has been elected president of the Mexican Mathematical Society.

Assistant Professor D. B. Ames of Rensselaer Polytechnic Institute has been promoted to an associate professorship.

Dr. K. J. Arnold of Massachusetts Institute of Technology has been appointed to an assistant professorship at the University of New Hampshire.

Assistant Professor Theodore Bennett of Marietta College has been promoted to a professorship.

Assistant Professor M. T. Bird of Utah State Agricultural College has been promoted to an associate professorship.

Dr. R. P. Boas, Jr., has been appointed visiting lecturer at Harvard University.

Assistant Professor W. M. Borgman, Jr., of Wayne University has been promoted to an associate professorship.

Assistant Professor Richard Brauer of the University of Toronto has been promoted to an associate professorship.

Assistant Professor F. J. H. Burkett of Union College has been promoted to an associate professorship.

Dr. Herbert Busemann of the Illinois Institute of Technology has been promoted to an assistant professorship.

Dr. Nathaniel Coburn of the University of Texas has been promoted to an assistant professorship.

Associate Professor A. H. Copeland of the University of Michigan has been promoted to a professorship.

Assistant Professor H. S. M. Coxeter of the University of Toronto has been promoted to an associate professorship.

Dr. John DeCicco of Illinois Institute of Technology has been promoted to an assistant professorship.

Assistant Professor D. G. Fulton of Ohio Northern University has been appointed to an assistant professorship at the University of New Hampshire.

Associate Professor W. H. Gage of the University of British Columbia has been promoted to a professorship.

Sister Agnes A. Green of the Immaculate Heart College, Los Angeles, California, has been promoted to an associate professorship.

Assistant Professor G. G. Harvey of the Massachusetts Institute of Technology has been promoted to an associate professorship.

Dr. F. B. Hildebrand of Massachusetts Institute of Technology has been promoted to an assistant professorship.

Dr. A. S. Householder of the University of Chicago has been promoted to an assistant professorship of mathematical biophysics.

Assistant Professor G. B. Huff of Southern Methodist University has been promoted to an associate professorship.

Professor R. D. James of the University of Saskatchewan has been appointed to a professorship at the University of British Columbia.

Assistant Professor Marie M. Johnson of Oberlin College has been promoted to an associate professorship.

Dr. Mark Kac of Cornell University has been promoted to an assistant professorship.

Associate Professor Claribel Kendall of the University of Colorado has been promoted to a professorship.

Professor C. C. MacDuffee of Hunter College has been appointed to a professorship at the University of Wisconsin.

Dr. Rhoda Manning of Oregon State College has been promoted to an assistant professorship.

Professor Deane Montgomery of Smith College is on leave at Princeton University.

Associate Professor C. A. Nelson of New Jersey College for Women, Rutgers University, has been promoted to a professorship.

Dr. H. E. Newell of the University of Maryland has been promoted to an assistant professorship.

Professor Abba V. Newton of Hartwick College, Oneonta, New York, has been appointed to an assistant professorship at Smith College.

Dr. E. P. Northrop of Hotchkiss School, Lakeville, Connecticut, has been appointed to an assistant professorship at the University of Chicago.

Assistant Professor Gordon Pall of McGill University has been promoted to an associate professorship.

Assistant Professor P. M. Pepper of the University of Notre Dame has been promoted to an associate professorship.

Dr. Harry Pollard of Harvard University has been appointed to an assistant professorship at Kenyon College, Gambier, Ohio.

Associate Professor G. B. Price of the University of Kansas has been promoted to a professorship.

Dr. J. K. Riess of Brown University has been appointed to an assistant professorship at Tulane University.

Professor L. D. Rodabaugh of Shurtleff College, Alton, Illinois, has been appointed visiting lecturer at Oberlin College.

Dr. Ralph Salem of the Massachusetts Institute of Technology has been promoted to an assistant professorship.

Dr. O. F. G. Schilling of the University of Chicago has been promoted to an assistant professorship.

Dr. Edith R. Schneckenburger of Michigan State Normal College has been promoted to an assistant professorship.

Assistant Professor A. J. Smith of the College of William and Mary has been appointed to an assistant professorship at Montana School of Mines.

Professor C. C. Spooner of the Northern Michigan College of Education has retired.

Assistant Professor B. M. Stewart of Michigan State College has been appointed to an assistant professorship at Denison University.

Assistant Professor I. D. Stewart of Whitman College, Walla Walla, Washington, has been promoted to an associate professorship.

Assistant Professor A. H. Taub of the University of Washington has been promoted to an associate professorship.

Mr. M. L. Vest of West Virginia University has been promoted to an assistant professorship.

Dr. Alexander Weinstein of the University of Toronto has been promoted to an assistant professorship.

Assistant Professor F. M. Wood of McGill University has been promoted to an associate professorship.

Dr. H. S. Zuckerman of the University of Washington has been promoted to an assistant professorship.

The following appointments to instructorships are announced:
Iowa State Teachers College: E. Marie Hove
Princton University: L. L. Rauch
Rice Institute: George Piranian
Syracuse University: F. L. Celauro
Vassar College: Janet C. Durand

Professor James McGiffert of Rensselaer Polytechnic Institute died June 18 1943, at the age of 75 years.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

SERVICE OPPORTUNITIES FOR WOMEN

From the Navy: The Navy is making an urgent search for women to perform important services in the Aerology and in the Electronic Programs. These Programs are under the sponsorship of the WAVES. In addition to the regular requirements for a candidate for officer training, the following qualifications are necessary. For Aerology, the applicant must have had college mathematics through the integral calculus, and also have had one year of physics. For Electronics, the requirement is one year of college mathematics and one year of college physics. College graduates who have majored in mathematics are especially urged to apply. After a period of training in the regular Officer Training Schools, persons enlisted in these Programs will continue with special work at various colleges and universities throughout the country. For further details, one should consult the nearest Office of Naval Officer Procurement.

From Civil Service: Persons who have completed a four-year college course with study in mathematics through the calculus may be eligible for Federal positions paying $2,533 a year. Mathematicians are needed in the Coast and

Geodetic Survey, in the Hydrographic Office and Naval Observatory, in the Navy Department, in the War Department arsenals, and by the National Advisory Committee for Aeronautics. Persons with one year of college and one course in mathematics may qualify for mathematical aid positions, paying $1,970 and $1,752 a year. Completion of a war training course in mathematics is qualifying for these latter positions. One year of appropriate experience is also qualifying. Higher training is required for positions as aids paying up to $3,163 a year. Mathematical aids are employed in the Department of Commerce and in the War and Navy Departments. Trainee aids in mathematics may be employed in positions existing in Washington, D. C., only. Persons who have completed a high school course in mathematics, biology, physics, chemistry, or general science are eligible for trainee positions, and may apply to take the written test. Appointees perform simple computations, and are paid at the rate of $1,560, $1,620, and $1,752 a year while learning the duties of the position. Junior astronomers in the Naval Observatory, Washington, D. C., assist in making astronomical observations or computations, preparing information for publication, and caring for instruments and equipment. For Junior Astronomer positions paying $2,433 a year, a four-year course in a recognized college, with a minimum of twelve semester hours in mathematics and twelve semester hours in astronomy, will qualify. Women are especially asked to apply for these various positions.

IMPORTANT BILLS BEFORE CONGRESS

The Victory Corps Act of 1943 (S. 875), sponsored by Senator Carl Hayden. This bill before the Congress of the United States has received the official support of the Army and the Navy, and has been favorably recommended by the Senate Committee on Education and Labor. The endorsement of the latter Committee resulted from such conclusions as the following: "The Army and the Navy have found glaring deficiencies in basic preparation in the fields of mathematics and science. They should not be called upon to take time out of the limited time available for training of soldiers to make up these deficiencies. The schools can be helped to do much in the way of revising courses and improving basic training in these fields."

In brief, the bill provides an authorization for an annual appropriation of $8,484,377 for the war service preinduction training of high-school students in mathematics, science, and preflight aeronautics, the selection of students to be prepared for war service, and the physical training of such students. The authorization is limited to the duration of the present war, and follows the usual pattern of Federal grants-in-aid to the States, the procedures for which have been developed over a period of some twenty-five years. This pattern gives great latitude to the States in the development of their own plans within the limitations of the purposes of the grants-in-aid, and is therefore flexible and adjustable to the different needs which exist in the several States. The bill does not set up a Federal system of preinduction education and training. It undertakes only to

encourage and to aid all the States in their own high-school programs of improvement to meet clearly defined and pressing war needs. A definite formula is provided for the allotments to the States of funds authorized.

The Science Mobilization Bill (S. 702), sponsored by Senator H. M. Kilgore. The necessity of the bill, as conceived by its sponsor, is explained in the preliminary part of the first section as follows: "The Congress hereby recognizes that the full development and application of the Nation's scientific and technical resources are necessary for the effective prosecution of the war and for peacetime progress and prosperity, and that serious impediments thereto consist in the unassembled and uncoordinated state of information concerning existing scientific and technical resources; the lack of an adequate appraisal, and the unplanned and improvident training, development, and use, of scientific and technical personnel, resources, and facilities in relation to the national need; the consequent delay and ineffectiveness in meeting the urgent scientific and technical problems of the national defense and essential civilian needs; the trend toward monopolized control of scientific and technical data and other resources with lack of access thereto in the public interest; and the absence of an effective Federal organization to promote and coordinate, in the national interest, scientific and technical developments."

To remove these "impediments," it is proposed to establish an "Office of Scientific and Technical Mobilization." The National Roster of Scientific and Specialized Personnel of the War Manpower Commission would be abolished as a separate entity, and its powers, personnel, and unexpended appropriation would be transferred to the new agency. The "Office" would be directed by an "Administrator," appointed by the President to serve at the pleasure of the President, with the assistance of a salaried "National Scientific and Technical Board," also appointed by the President, but whose duties are to be defined by the Administrator. This Board is to consist of six members besides the Administrator, one to represent industry, one to represent agriculture, one to represent labor, one to represent the consuming public, and two additional members at large who shall be scientists or technologists. The Administrator will appoint all other employees, and he may waive the provisions of the civil-service laws and regulations if he determines that it is necessary to do so. Provision is also made for a non-salaried "National Scientific and Technical Committee," to be appointed by the President to advise and consult with the Administrator at least once a month upon basic policies of the Board. This Committee consists of the Board previously mentioned, a representative from each of such Federal departments as the President shall designate, four additional representatives of the consuming public, six additional members representing labor, six representing management, and three additional scientists or technologists.

The bill vests the Office with broad powers over "scientific and technical personnel" and "scientific and technical facilities." By definition, "Scientific and technical facilities shall include all real and personal property, . . . programs, projects, . . . methods, processes, procedures, . . . patents, inventions, . . . in-

formation or knowledge of every description used or intended to be used for scientific or technical purposes in research and development or in the production or supply of war or civilian goods or services." "Scientific and technical personnel shall include all persons, excepting physicians and dentists, who have completed any course of study in any college or university in any branch of science or its practical application or who have had not less than an aggregate of six months' training or employment in any scientific or technical vocation."

During August, the Executive Committee of the American Association for the Advancement of Science passed the following resolution pertaining to the bill: "After careful consideration of the purposes and provisions of the Science Mobilization Bill (S.702), the American Association for the Advancement of Science, an organization having nearly 25,000 members (and having 187 associated and affiliated societies with a combined membership of over 500,000 persons whose interests cover broadly all the natural and social sciences) now, through its Council of about 250 members chosen from among the leaders of American science, respectfully recommends to the Senate and to the House of Representatives of the United States that the Kilgore Bill (S.702) be not passed either in its present form or in any other form containing similar provisions."

FROM SELECTIVE SERVICE

Student Deferment. The provisions of Bulletin No. 33-6 pertaining to the deferment of undergraduate students were revised on July 24. The revised statement follows.

"A student in undergraduate work in any of the scientific and specialized fields . . . should be considered for occupational classification if he is a full-time student in good standing in a recognized college or university and if it is certified by the institution as follows:

(a) That he is competent and gives promise of successful completion of such course of study, and

(b) That if he continues his progress he will graduate from such a course of study within 24 months from the date of certification."

New Occupational Deferment Policy. Local Board Memorandum No. 115, as amended August 16, 1943, sets forth a new occupational deferment policy through the medium of an extensive list of critical occupations (I: Production and Services Occupations; II: Professional and Scientific Occupations), and a list of nondeferrable activities and occupations. The occupation, "Mathematician (including Cryptanalyst)," is listed in Part II of the first list. The Memorandum specifically states that the titles appearing in this critical list "are also intended to cover those persons who are engaged in full-time teaching of these professions." The relation of the new memorandum to previous Activity and Occupation Bulletins is clarified by the sentence, "The Activity and Occupation Bulletins should be used by the agencies of the Selective Service System as a guide and should be considered in occupational classification matters along with all other available information."

COMMITTEE ON AVAILABLE TEACHERS OF COLLEGIATE MATHEMATICS

The Committee on Available Teachers of Collegiate Mathematics, established by the War Policy Committee of the American Mathematical Society and the Mathematical Association of America, has been in existence since the beginning of April, 1943. During this time it has received and answered numerous inquiries from colleges and universities needing teachers of mathematics, as well as from teachers who were free to accept appointments.

A total of 149 persons have registered with the Committee. Of this number, 129 have at least the master's degree. The Committee has answered requests for teachers of mathematics from 67 different institutions. To these institutions a total of 348 names, representing 111 different persons, have been suggested. According to the records of the Committee, 57 of the persons registered are no longer available. It should be pointed out that of the remaining candidates some have indicated that they are available only for summer or part-time teaching, others have had only secondary school teaching experience, while others are available for appointment in restricted geographical areas.

It is anticipated that the demand for teachers will increase considerably during the next two or three months. On the other hand, the number of available well-qualified candidates who have registered with the Committee for such appointments has been reduced to such an extent that the remaining supply has become quite inadequate to meet the expected demand.

For this reason, the committee requests that (a) individual teachers report their availability and (b) departments of mathematics inform the Committee at the earliest possible date of their needs during the next half year, giving as full details as they can concerning the qualifications expected, the salary offered and other pertinent facts relating to their vacancies, and of members who are free to fill temporary positions in other institutions.

The effectiveness of the work of this Committee will be enhanced if registrants will report positions which they have accepted and if inquiring institutions will report appointments that have been made.

Committee on Available Teachers,
W. D. CAIRNS
ARNOLD DRESDEN
J. R. KLINE

110 Bennett Hall
University of Pennsylvania
September 2, 1943.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE SPRING MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The nineteenth meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Pennsylvania College for Women, Pittsburgh, Pennsylvania, on April 17, 1943. The chairman of the Section, Dr. R. G. Sturm of the Aluminum Research Laboratories, presided at both morning and afternoon sessions.

The attendance was twenty-four, including the following eight members of the Association: Helen Calkins, H. L. Dorwart, B. P. Hoover, David Moskovitz, E. G. Olds, J. B. Rosenbach, R. G. Sturm, E. D. Wells.

At a business meeting prior to the afternoon session, the present officers were reelected for another term. It was voted to hold the next meeting during the month of April, 1944. The invitation of Carnegie Institute of Technology to hold the meeting at that institution was accepted.

After an address of welcome by Dean Helen Marks of Pennsylvania College for Women, the following papers were presented:

1. *New theorems on ordinary linear differential equations*, by Dr. R. F. Clippinger, Carnegie Institute of Technology, introduced by Professor Dorwart.

The speaker presented an expository treatment of certain recent developments of his own generalizing Sturm-Liouville theorems in several directions.

2. *Application of mathematics to some geophysical problems*, by Dr. Sigmund Hammer, Gulf Research Laboratories, introduced by Dr. Sturm.

Dr. Hammer discussed applications of mathematics to problems in geophysical exploration. The subject was introduced by a description of the technique of prospecting for minerals by measuring exceedingly small irregularities ("anomalies") in the earth's gravitational field caused by buried mineral masses or deformed rocks. The method requires a precise knowledge of the normal gravitational field of the earth. The speaker discussed the effect upon the earth's gravitational field of the centripetal acceleration due to the rotation of the earth, and also the effect due to the spheroidal shape of the earth. He then considered the application of a theorem of Gauss to the problem of estimating the mass of a buried body of ore from its gravitational anomaly. The paper was concluded with a brief mention of important applications of other mathematical concepts to geophysical problems.

3. *Approximations for complex roots of algebraic equations*, by Professor H. C. Hicks, Carnegie Institute of Technology, introduced by Professor Dorwart.

This paper dealt with a practical method of approximating complex roots of algebraic equations. The method was compared with other standard devices such as that of Graeffe.

4. *A mathematical peculiarity of the plastic stress-strain relations*, by E. A. Davis, Westinghouse Research Laboratories, introduced by Dr. Sturm.

Mr. Davis stated that when a ductile metal flows under the influence of combined stresses, three mutually perpendicular directions can be found along which the deformations are pure extensions. There are two theories at present which deal with the values of the three principal strain rates at any given instant. The older is based upon the law of viscous flow which states that the shear rate on any plane of principal shear stress is proportional to the stress acting on that plane. The newer theory claims that the rates are not proportional to the stresses, but that they may be expressed by relations involving power functions of the stresses. This theory reduces to the older one when a certain exponent n has the value 1. The peculiarity pointed out is that in the newer theory the distribution of strain rates when $n=1$ is the same as the distribution when $n=3$. This is due to the fact that the expression $[a^n-b^n]/[(a+b)^n+b^n]$ has the same value for $n=3$ and for $n=1$.

H. L. DORWART, *Secretary*

THE ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The twenty-seventh annual meeting of the Rocky Mountain Section of the Mathematical Association of American was held on Friday and Saturday, April 16–17, 1943, at the University of Denver, Denver, Colorado. It was a joint meeting with the National Council of Teachers of Mathematics and the Eastern Division of the Colorado Education Association. Section meetings were held Friday afternoon and evening, at both of which the Vice-Chairman of the Section, Professor A. W. Recht, presided. Three additional sessions were held on Saturday in conjunction with the other organizations participating in the meeting.

The attendance was one hundred and thirty-five, including the following fifteen members of the Association: A. G. Clark, Sister Rose Margaret Cook, J. R. Everett, J. C. Fitterer, G. W. Gorrell, D. F. Gunder, J. O. Hassler, A. J. Kempner, Claribel Kendall, W. J. LeVeque, A. J. Lewis, A. E. Mallory, A. W. Recht, C. H. Sisam, H. W. Williams.

At the business meeting the following officers were elected for the coming year: Chairman, Professor A. E. Mallory, Colorado State College of Education; Vice-Chairman, Professor A. J. Kempner, University of Colorado.

The following papers were presented:

1. *On the place of mechanics in the system of sciences, and the training of mathematicians for work in applied mechanics*, by Dr. Paul Nemenyi, University of Colorado, introduced by Professor Kempner.

The speaker discussed the unity of the sciences and the place of mechanics in the scheme of scientific studies. He classified mechanics as a part of physics, and considered the relation of mechanics to other parts of physics and to mathe-

matics. In particular, the relation between mechanics and probability was brought out. A program for the training of mathematicians for research and teaching in mechanics was then outlined.

2. *A method of measuring effectiveness in the teaching of college mathematics*, by Professor J. O. Hassler, University of Oklahoma.

Professor Hassler investigated the grades in Calculus II of the students of eleven teachers of Calculus I. The records of the "A," "B," "C" and "D" students were examined separately, and the successes (by separate groups) of the students of the various teachers compared. This was a measure of the effectiveness of the teaching in the first course. The students in each of the four grade classifications were also divided into two groups, namely those who remained with the same teacher and those who had the second course with a different teacher. In this way was obtained an evaluation of the teachers' grading scales.

3. *The integral* $\int x^{-1}dx = \log x$ *as a limiting case of* $\int x^{n-1}dx = x^n/n$, by W. J. LeVeque, University of Colorado.

In this paper it was shown that the integral $\int x^{-1}dx$ can be studied by considering the limit of $\int x^{n-1}dx$ as n approaches zero. The geometric properties of the approximation curves were also investigated.

4. *On a continuous stochastic process*, by Professor A. G. Clark, Colorado State College.

Professor Clark stated that, in ballistics research, it has been the practice to measure the variability of an ordered succession of random variables $x_1, x_2, \cdots, x_n$ by using the mean square successive difference $\delta^2 = (N-1)^{-1}\sum_{i=1}^{N-1}(x_{i+1}-x_i)^2$ as a criterion for measuring variability, rather than the quantity $s^2 = N^{-1}\sum_{i=1}^{N}(x_i-\bar{x})^2$. He considered the problem of testing for determination of trend, and showed that δ/s is the proper criterion to use for this purpose.

5. *On the introduction of coördinates in an affine plane geometry*, by Mrs. Margaret S. Matchett, University of Denver, introduced by Professor Recht.

Using only axioms of connection for points and lines, the parallel axiom, and the configuration of Desargues, it is possible to introduce a system of coördinates into a geometry. These coördinates satisfy all the field properties except that of commutative multiplication. In order to define these coördinates one considers the group of those one-to-one transformations of the plane which map parallel lines into parallel lines. The translations form an invariant sub-group of this group. The inner automorphisms of this sub-group, with addition and multiplication suitably defined, form the division algebra from which the coördinates are taken.

6. *Graphical methods for representation of various types of functional relations*, by Professor A. J. Kempner, University of Colorado.

This paper dealt with a method of plotting the graph of an equation of the type $F[f(x, y), g(x, y)] = C$. The details are as follows: Let $X = f(x, y)$ and

$Y=g(x, y)$. Plot the curve $F(X, Y)=C$ with reference to X and Y axes, and plot on a separate chart the two families of curves $f(x, y)=\alpha$ and $g(x, y)=\beta$ where α and β are parameters. If (α, β) is any point on the curve $F(X, Y)=C$, a point of intersection of the two curves $f(x, y)=\alpha$ and $g(x, y)=\beta$ is a point of the curve $F[f(x, y), g(x, y)]=C$. The method can be extended to the representation of the equation $F[f(x, y), g(x, y)]=H(z)$, leading to a one parameter family of curves in the xy-plane.

7. *Teaching mathematics effectively for war or peace*, by Professor J. O. Hassler, University of Oklahoma.

Professor Hassler reviewed briefly the controversy over the transfer of training, and reported that eighty per cent of the psychological experiments up to date show clear evidence of transfer of training. He remarked that to teach consciously for transfer of training is a prime goal of effective teaching. It was also stated that this object can be achieved by relating subject-matter in every possible way to practical applications, by cultivating habits of independent thinking and generalization, and by exploiting the spirit of discovery in the pupil.

8. *Mathematics abridged has gone to war*, by Professor J. O. Hassler, University of Oklahoma.

The speaker reviewed the present situation wherein frantic efforts (by means of concentrated courses) are being made to make amends for deficient training in mathematics among the youths in the army or about to go into the army. He gave some facts concerning dilution of mathematics courses in the recent past which has contributed to this delinquency. He made a plea for teachers to equip themselves to fight against having a denatured, abridged, and diluted mathematics in the post-war curriculum of the high schools.

A. J. LEWIS, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-third regular meeting of the Southern California Section of the Mathematical Association of America was held at the University of Southern California, Los Angeles, California, on Saturday, March 13, 1943. Professor Morgan Ward, chairman of the Section, presided.

The attendance was fifty-five, including the following twenty-six members of the Association: O. W. Albert, C. K. Alexander, L. D. Ames, Clifford Bell, L. T. Black, Myrtie Collier, P. H. Daus, D. C. Duncan, W. H. Glenn, Jr., Frances C. Hinds, P. G. Hoel, C. G. Jaeger, G. R. Kaelin, Ada A. McClellan, G. F. McEwen, P. M. Niersbach, W. T. Puckett, Jr., H. R. Pyle, J. M. Robb, G. E. F. Sherwood, D. V. Steed, A. E. Taylor, S. E. Urner, Morgan Ward, W. M. Whyburn, Euphemia R. Worthington.

The following officers were elected for the coming year: Chairman, D. C. Duncan, Los Angeles City College; Vice-Chairman, P. G. Hoel, University of California at Los Angeles; Program Committee, C. K. Alexander, Chairman, P. M. Niersbach, and the Secretary. The next meeting was tentatively scheduled to be held March 11, 1944, at Los Angeles City College.

The following six papers were read:

1. *The slide rule—modern uses and devices*, by Dr. E. J. Hills, Los Angeles City College, introduced by Professor Urner.

Dr. Hills described the range of usefulness of the rule and the types of slide rules available, and showed how new scales can be and are developed. He considered the most natural ways to multiply, divide, obtain squares, and extract square roots. He stressed the importance of the principle of proportion and urged that it be used as much as possible. In conclusion, he discussed the use of the *LL* scales, and included an analysis of the natural functions.

2. *Training of the naval aviation cadet*, by Lieutenant J. L. Brader, U. S. N., introduced by Professor Ward.

3. *Report of the subcommittee on mathematics of the California committee on education*, by Professor W. M. Whyburn, University of California at Los Angeles.

Professor Whyburn, chairman of the California subcommittee on mathematics, gave a brief discussion of the organization of this subcommittee and of the chief problems which it has under study. He then read a portion of a preliminary report made by the subcommittee in October, and this portion of the report was endorsed by the Section. This same portion of the report had been presented to the Northern California Section, and was endorsed by that body. That part of the report which was read by Professor Whyburn was printed in this MONTHLY, vol. 50, 1943, p. 337.

4. *The application of geometrical probability to sampling problems in ecology*, by Professor P. G. Hoel, University of California at Los Angeles.

The speaker treated the problem of sampling a region for coverage or abundance of a given species of plants from a geometrical point of view. He applied statistical distribution theory and geometrical probability to obtain formulas which measure the relative accuracy of various common sampling techniques in ecology.

5. *Arithmetical properties of recurring series*, by Professor Morgan Ward, California Institute of Technology.

This paper dealt with the sequence of integers (u) defined recursively by the equality

$$u_{n+k} = p_1 u_{n+k-1} + p_2 u_{n+k-2} + \cdots + p_k u_n$$

where $p_1, p_2, \cdots, p_k$ are given integers, and the initial values of (u) are also integers. If $u_0 = 0$, $u = 1$, and if u_n divides u_m whenever n divides m, then (u) is

called a Lucasian sequence in honor of Lucas, who studied the case $k=2$. Professor Ward obtained new results for the cases $k=4$, 5, and 6. In particular, he showed that if $k=5$, the polynomial $f(x)$ associated with the sequence (u) must have an integral root; and that if $k=4$, and $f(x)$ is irreducible, its Galois group is of order 2, 4, or 8.

6. *Certain generalizations of the formulae for the range of normally distributed variates*, by Professor G. F. McEwen, Scripps Institution of Oceanography.

By substitution of logarithms of the variates in existing formulae for the range R of a normally distributed variate, the speaker obtained more general formulae for this special type of asymmetry. One result was

$$R = \sinh\left[\frac{a_n}{2}\sigma_{\ln x}\right]$$

where a_n is a known function of variates. Similarly, from a formula relating the range R_1 to R_{rn}, the mean difference between the rn greatest and the rn smallest variates, and the assumption of a normal distribution, he derived general formulae for the asymmetrical case.

P. H. DAUS, *Secretary*

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-seventh Annual Meeting, Chicago, Illinois, November 27–28, 1943.

The following is a list of the Sections of the Associations with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April, 1944
ILLINOIS
INDIANA, Indianapolis, Oct. 29–30, 1943
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, Jan. 29, 1944
OHIO, Columbus, April 6, 1944
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 11, 1944
SOUTHWESTERN
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As determined more recently by the Trustees, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included; they carry their own reward.

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